

An FPTAS for the Volume of a \mathcal{V} -polytope —It is Hard to Compute The Volume of The Intersection of Two Cross-polytopes

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Abstract

Given an n -dimensional convex body by a membership oracle in general, it is known that any polynomial-time *deterministic* algorithm cannot approximate its volume within ratio $(n/\log n)^n$. There is a substantial progress on *randomized* approximation such as Markov chain Monte Carlo for a high-dimensional volume, and for many #P-hard problems, while some deterministic approximation algorithms are recently developed only for a few #P-hard problems. Motivated by a *deterministic* approximation of the volume of a \mathcal{V} -polytope, that is a polytope with few vertices and (possibly) exponentially many facets, this paper investigates the volume of a “knapsack dual polytope,” which is known to be #P-hard due to Khachiyan (1989). We reduce an approximate volume of a knapsack dual polytope to that of the *intersection of two cross-polytopes*, and give FPTASs for those volume computations. Interestingly, the volume of the intersection of two cross-polytopes (i.e., L_1 -balls) is #P-hard, unlike the cases of L_∞ -balls or L_2 -balls.

Keywords: Deterministic approximation, #P-hard, \mathcal{V} -polytope, intersection of L_1 -balls

1 Introduction

1.1 Approximation of a high dimensional volume: randomized vs. deterministic

A high dimensional volume is hard to compute, even for approximation. When an n -dimensional convex body is given by a *membership oracle*, no polynomial-time *deterministic* algorithm can approximate its volume within ratio $(n/\log n)^n$ [3, 10, 20, 6]. Intuitively, the impossibility comes from the fact that the volume of an n -dimensional L_∞ -ball (i.e., hypercube) is exponentially large to the volume of its inscribed L_2 -ball or L_1 -ball, nevertheless the L_2 -ball (L_1 -ball as well) is convex and touches each facet of the L_∞ -ball (see e.g., [22]). Lovász said in [20] for a convex body K that “If K is a polytope, then there may be much better ways to compute $\text{Vol}(K)$.” Unfortunately, an exact volume is often #P-hard, even for a relatively simple polytope. For instance, the *volume* of a knapsack polytope, which is given by a box constraint (i.e., hypercube $[0, 1]^n$) and a single linear inequality, is a well-known #P-hard problem [8].

The difficulty caused by the exponential gap between L_∞ -ball and L_1 -ball also does harm a simple Monte Carlo algorithm. Then, the Markov chain Monte Carlo (MCMC) method, a sophisticated *randomized* algorithm, achieves a great success for approximating a high volume. Dyer, Frieze and Kannan [9] gave the first fully polynomial-time randomized approximation scheme (FPRAS) for the volume computation of

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a general convex body¹. They employed a *grid-walk*, which is efficiently implemented with a membership oracle, and showed its rapidly mixing, then they gave an FPRAS runs in $O^*(n^{23})$ time where O^* ignores $\text{poly}(\log n)$ and $1/\epsilon$ terms. After several improvements, Lovász and Vempala [21] improved the time complexity to $O^*(n^4)$ in which they employ hit-and-run walk, and recently Cousins and Vempala [5] gave an $O^*(n^3)$ -time algorithm. Many randomized techniques, including MCMC, also have been developed for designing FPRAS for #P-hard problems.

In contrast, a development of a *deterministic* approximation for #P-hard problems is a current challenge, and not many results seem to be known. A remarkable progress is the *correlation decay* argument due to Weitz [24]; he designed a *fully polynomial time approximation scheme (FPTAS)* for counting independent sets in graphs whose maximum degree is at least 5. A similar technique is independently presented by Bandyopadhyay and Gamarnik [2], and there are several recent developments on the technique, e.g., [11, 4, 16, 17, 19]. For counting knapsack solutions², Gopalan, Klivans and Meka [12], and Štefankovič, Vempala and Vigoda [23] gave deterministic approximation algorithms based on the dynamic programming (see also [13]), in a similar way to a simple random sampling algorithm by Dyer [7]. Modifying the dynamic programming, Li and Shi [18] gave an FPTAS for the volume of a knapsack polytope, which runs in $O((n^3/\epsilon^2)\text{poly log } b)$ time where b is the capacity of a knapsack. Motivated by a different approach, Ando and Kijima [1] gave another FPTAS for the volume of a knapsack polytope. Their scheme is based on a classical approximate convolution, and runs in $O(n^3/\epsilon)$ time, independent of the size of items and the capacity of a knapsack reckoning without numerical calculus.

1.2 \mathcal{H} -polytope and \mathcal{V} -polytope

An \mathcal{H} -polyhedron is an intersection of finitely many closed half-spaces in \mathbb{R}^n . An \mathcal{H} -polytope is a bounded \mathcal{H} -polyhedron. A \mathcal{V} -polytope is a convex hull of a finite point set in \mathbb{R}^n [22]. From the view point of computational complexity, a major difference between an \mathcal{H} -polytope and a \mathcal{V} -polytope is the measure of their ‘input size.’ An \mathcal{H} -polytope given by linear inequalities defining half-spaces may have vertices exponentially many to the number of the inequalities, e.g., an n -dimensional hypercube is given by $2n$ linear inequalities as an \mathcal{H} -polytope, and has 2^n vertices. In contrast, a \mathcal{V} -polytope given by a point set may have facets exponentially many to the number of vertices, e.g., an n -dimensional cross-polytope (that is an L_1 -ball, in fact) is given by a set of $2n$ points as a \mathcal{V} -polytope, and it has 2^n facets.

There are many interesting properties, that are known, or unknown, between \mathcal{H} -polytope and \mathcal{V} -polytope [22]. A membership query is polynomial time for both \mathcal{H} -polytope and \mathcal{V} -polytope. It is still unknown about the complexity of a query if a given pair of \mathcal{V} -polytope and \mathcal{H} -polytope are identical. Linear programming (LP) on a \mathcal{V} -polytope is trivially polynomial time since it is sufficient to check the objective value of all vertices and hence LP is usually concerned with an \mathcal{H} -polytope.

1.3 Volume of \mathcal{V} -polytope

Motivated by a hardness of the volume computation of a \mathcal{V} -polytope, Khachiyan [14] is concerned with the following \mathcal{V} -polytope: Suppose a vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$ is given, where without loss of generality we may assume that $a_1 \geq a_2 \geq \dots \geq a_n$. Then let

$$P_{\mathbf{a}} \stackrel{\text{def}}{=} \text{conv} \{ \pm \mathbf{e}_1, \dots, \pm \mathbf{e}_n, \mathbf{a} \} \quad (1)$$

¹ Precisely, they are concerned with a “well-rounded” convex body, after an affine transformation of a general finite convex body.

² Given $\mathbf{a} \in \mathbb{Z}_{\geq 0}^n$ and $b \in \mathbb{Z}_{>0}$, the problem is to compute $|\{ \mathbf{x} \in \{0, 1\}^n \mid \sum_{i=1}^n a_i x_i \leq b \}|$. Remark that it is computed in polynomial time when all the inputs a_i ($i = 1, \dots, n$) and b are bounded by $\text{poly}(n)$, using a version of the standard dynamic programming for knapsack problem (see e.g., [7, 13]). Nevertheless, it should be worth noting that [12] and [23] needed special techniques, different from ones for optimization problems, to design FPTASs for the counting problem.

where e_1, \dots, e_n are the standard basis vectors in \mathbb{R}^n . This paper calls $P_{\mathbf{a}}$ *knapsack dual polytope*³. Khachiyan [14] showed that computing $\text{Vol}(P_{\mathbf{a}})$ is $\#P$ -hard⁴. The hardness is given by a Cook reduction from counting set partitions, of which the decision version is a celebrated *weakly* NP-hard problem. We do not know any (efficient) technique to translate the volume between them a polytope and its dual polytope.

1.4 Contribution

Motivated by a development of techniques for *deterministic* approximation of the volumes of \mathcal{V} -polytopes, this paper investigates the knapsack dual polytope $P_{\mathbf{a}}$ given by (1). The main goal of the paper is to establish the following theorem.

Theorem 1.1. *For any ϵ ($0 < \epsilon < 1$), there exists a deterministic algorithm that outputs a value \hat{V} satisfying $(1 - \epsilon)\text{Vol}(P_{\mathbf{a}}) \leq \hat{V} \leq (1 + \epsilon)\text{Vol}(P_{\mathbf{a}})$ in $O(n^{10}\epsilon^{-6})$ time.*

As far as we know, this is the first result on designing an FPTAS for the volume of a \mathcal{V} -polytope which is known to be $\#P$ -hard. We also discuss some topics related to the volume of \mathcal{V} -polytopes appearing in the proof process. Let us briefly explain the outline of the paper.

Technique/organization The first step for Theorem 1.1 is a transformation of the *approximation problem* to another one: An approximate volume of $P_{\mathbf{a}}$ is reduced to the volume of a union of geometric sequence of cross-polytopes (Section 3.1), and then it is reduced to the volume of the intersection of two cross-polytopes (Section 3.2). We remark that the former reduction is just for approximation, and is useless for a $\#P$ -hardness. A technical point of this step is that the latter reduction is based on a subtraction—if you are familiar with an approximation, you may worry that a subtraction may destroy an approximation ratio⁵. It requires careful tuning of a parameter (β in Section 3) which plays conflicting functions in Sections 3.1 and 3.2: the larger β , the better approximation in Section 3.1, while the smaller β , the better in Section 3.2. Then, Section 3.3 claims by giving an appropriate β that if we have an FPTAS for the volume of an *intersection of two cross-polytopes* then we have an FPTAS of $\text{Vol}(P_{\mathbf{a}})$.

Section 4 is a technical core of the paper, where we give an FPTAS for the volume of the intersection of two cross-polytopes (i.e., L_1 -balls). The scheme is based on a modified version of the technique developed in [1], which is based on a classical approximate convolution. At a glance, the volume of the intersection of two-balls may seem easy. It is true for two L_∞ -balls (i.e., hypercubes⁶), or L_2 -balls (i.e., Euclidean balls). However, we show in Section 5 that the volume of the intersection of cross-polytopes is $\#P$ -hard. Intuitively, this interesting fact may come from the fact that the \mathcal{V} -polytope, meaning that an n -dimensional cross-polytope, has 2^n facets. In Section 6, we extend the technique in Section 4 to the intersection of any constant number of cross-polytopes. Section 7 briefly discusses the complexity of the volume computation of a \mathcal{V} -polytope regarding the number of vertices.

³ See [22] for the duality of polytopes. In fact, $P_{\mathbf{a}}$ itself is not the dual of a knapsack polytope in a canonical form, but it is obtained by an affine transformation from a dual of knapsack polytope under some assumptions. Khachiyan [15] says that computing $\text{Vol}(P_{\mathbf{a}})$ ‘is “polar” to determining the volume of the intersection of a cube and a halfspace.’

⁴ If all a_i ($i = 1, \dots, n$) are bounded by $\text{poly}(n)$, it is computed in polynomial time, so did the counting knapsack solutions. See also footnote 1 for counting knapsack solutions.

⁵ Suppose you know that x is approximately 49 within 1% error. Then, you know that $x + 50$ is approximately 99 within 1% error. However, it is difficult to say $50 - x$ is approximately 1. Even when additionally you know that x does not exceed 50, $50 - x$ may be 2, 1, 0.1 or smaller than 0.001, meaning that the approximation ratio is unbounded.

⁶ To be precise, an L_∞ -ball is a hypercube in a position parallel to the axis, meaning that any L_∞ -ball is transformed to any other one by scaling and parallel move, without using a rotation. If two hypercubes are not in a parallel position, the volume of the intersection is $\#P$ -hard since the volume of a knapsack polytope is.

2 Preliminary

This section presents some notation. Let $\text{conv}(S)$ denote the convex hull of $S \subseteq \mathbb{R}^n$, where S is not restricted to a finite point set. A *cross-polytope* $C(\mathbf{c}, r)$ of radius $r \in \mathbb{R}_{>0}$ centered at $\mathbf{c} \in \mathbb{R}^n$ is given by

$$C(\mathbf{c}, r) \stackrel{\text{def}}{=} \text{conv}\{\mathbf{c} \pm r\mathbf{e}_i \mid i = 1, \dots, n\} \quad (2)$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the standard basis vectors in \mathbb{R}^n . Clearly, $C(\mathbf{c}, r)$ has $2n$ vertices. In fact, $C(\mathbf{c}, r)$ is an L_1 -ball in \mathbb{R}^n described by

$$C(\mathbf{c}, r) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{c}\|_1 \leq r\} \quad (3)$$

$$= \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x} - \mathbf{c}, \boldsymbol{\sigma} \rangle \leq r \ (\forall \boldsymbol{\sigma} \in \{-1, 1\}^n)\} \quad (4)$$

where $\|\mathbf{u}\|_1 = \sum_{i=1}^n |u_i|$ for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n u_i v_i$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Note that $C(\mathbf{c}, r)$ has 2^n facets. It is not difficult to see that the volume of a cross-polytope in n -dimension is

$$\text{Vol}(C(\mathbf{c}, r)) = \frac{2^n}{n!} r^n \quad (5)$$

for any $r \geq 0$ and $\mathbf{c} \in \mathbb{R}^n$, where $\text{Vol}(S)$ for $S \subseteq \mathbb{R}^n$ denotes the (n -dimensional) volume of S .

3 FPTAS for Knapsack Dual Polytope

This section reduces an approximation of $\text{Vol}(P_{\mathbf{a}})$ to that of the intersection of two cross-polytopes. In Section 4, we will give an FPTAS for the volume of a latter polytope, accordingly we obtain Theorem 1.1.

3.1 Reduction to a geometric series of cross-polytopes

Let β be a parameter⁷ satisfying $0 < \beta < 1$, and let Q_0, Q_1, Q_2, \dots be a sequence of cross-polytopes defined by

$$Q_k \stackrel{\text{def}}{=} C((1 - \beta^k)\mathbf{a}, \beta^k) \quad (6)$$

for $k = 0, 1, 2, \dots$. Remark that

$$\begin{aligned} Q_0 &= C(\mathbf{0}, 1), \\ Q_1 &= C((1 - \beta)\mathbf{a}, \beta), \\ Q_\infty &= C(\mathbf{a}, 0) = \{\mathbf{a}\}. \end{aligned}$$

The goal of Section 3.1 is to establish the following.

Lemma 3.1. *Let ϵ satisfy $0 < \epsilon < 1$. If $1 - \beta \leq \frac{c_1 \epsilon}{n \|\mathbf{a}\|_1}$ where $0 < c_1 \epsilon < 1$, then*

$$(1 - c_1 \epsilon) \text{Vol}(P_{\mathbf{a}}) \leq \text{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) \leq \text{Vol}(P_{\mathbf{a}}).$$

We remark that $P_{\mathbf{a}}$ defined by (1) is also described by

$$P_{\mathbf{a}} = \text{conv}(C(\mathbf{0}, 1) \cup \{\mathbf{a}\}) \quad (7)$$

using $C(\mathbf{0}, 1)$. Figure 1 illustrates the approximation of $P_{\mathbf{a}}$ by this infinite sequence of cross-polytopes. The second inequality in Lemma 3.1 is relatively easy by the following lemma.

⁷ We will set $\beta = 1 - \frac{\epsilon}{2n \|\mathbf{a}\|_1}$, later.

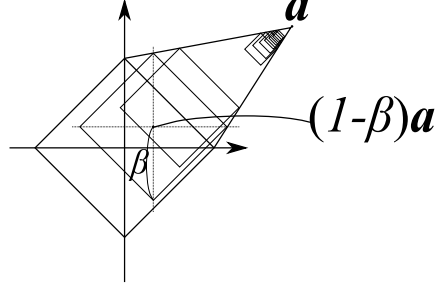


Figure 1: Approximating $P_{\mathbf{a}}$ by an infinite sequence of cross-polytopes.

Lemma 3.2.

$$\bigcup_{k=0}^{\infty} Q_k \subseteq P_{\mathbf{a}}.$$

Proof. Notice that $P_{\mathbf{a}}$ is the convex hull of Q_0 and \mathbf{a} (see (1) for the definition of $P_{\mathbf{a}}$). We give a map $\eta_k: Q_k \rightarrow Q_0$, and show that any $\mathbf{x} \in Q_k$ is in the line segment between $\eta_k(\mathbf{x})$ and \mathbf{a} . Notice that both Q_k and Q_0 are L_1 -balls, meaning that they are similar, and our map η_k is a natural correspondence between them. Let

$$\eta_k(\mathbf{x}) = \frac{\mathbf{x} - (1 - \beta^k)\mathbf{a}}{\beta^k}. \quad (8)$$

Then, $\|\eta_k(\mathbf{x}) - \mathbf{0}\|_1 \leq 1$ holds since $\|\mathbf{x} - (1 - \beta^k)\mathbf{a}\|_1 \leq \beta^k$ by the assumption that $\mathbf{x} \in Q_k$ (recall the definition (6) of Q_k). This implies that $\eta_k(\mathbf{x}) \in Q_0$. Notice that (8) implies that

$$\mathbf{x} = \beta^k \eta_k(\mathbf{x}) + (1 - \beta^k)\mathbf{a}$$

where $0 < \beta^k < 1$, meaning that \mathbf{x} is given by a convex combination of $\eta_k(\mathbf{x})$ and \mathbf{a} . \square

Next, we show the first inequality in Lemma 3.1. As a preliminary, we show the following.

Lemma 3.3.

$$\bigcup_{k=0}^{\infty} \text{conv}(Q_k \cup Q_{k+1}) \cup \{\mathbf{a}\} \supseteq P_{\mathbf{a}}.$$

In fact, Lemma 3.3 holds by equality, but we only show \supseteq here.

Proof. Suppose $\mathbf{x} \in P_{\mathbf{a}}$. Since $P_{\mathbf{a}} = \text{conv}(C(\mathbf{0}, 1) \cup \{\mathbf{a}\})$, it is not difficult to see that there is a $\mathbf{y}_0 \in C(\mathbf{0}, 1)$ such that \mathbf{x} is in the line segment $\overline{\mathbf{a}, \mathbf{y}}$ between \mathbf{a} and \mathbf{y} . Using bijective maps η_k for $k = 1, 2, \dots$ defined by (8), $\eta_k^{-1}(\mathbf{y})$ is in $\overline{\mathbf{a}, \mathbf{y}}$ in order of k . Suppose \mathbf{x} is between $\eta_{k^*}^{-1}(\mathbf{y})$ and $\eta_{k^*+1}^{-1}(\mathbf{y})$, then $\mathbf{x} \in \text{conv}(Q_{k^*} \cup Q_{k^*+1})$. We obtain the claim. \square

Lemma 3.4. If $1 - \beta \leq \frac{c_1 \epsilon}{n \|\mathbf{a}\|_1}$, then

$$\text{Vol} \left(\bigcup_{k=0}^{\infty} Q_k \right) \geq (1 - c_1 \epsilon) \text{Vol}(P_{\mathbf{a}}).$$

Proof. For convenience, let

$$\begin{aligned}\Delta &\stackrel{\text{def}}{=} (1 - \beta)\|\mathbf{a}\|_1 \\ &\leq \frac{c_1\epsilon}{n\|\mathbf{a}\|_1}\|\mathbf{a}\|_1 = \frac{c_1\epsilon}{n}\end{aligned}\tag{9}$$

Notice that Δ is the distance between the centers of cross-polytopes Q_0 and Q_1 . Let

$$R_k \stackrel{\text{def}}{=} C((1 - \beta^k)\mathbf{a}, (1 + \Delta)\beta^k)$$

e.g., $R_0 = C(\mathbf{0}, 1 + \Delta)$. We claim that $R_k \supseteq \text{conv}(Q_k \cup Q_{k+1})$, which implies $\bigcup_{k=0}^{\infty} R_k \cup \{\mathbf{a}\} \supseteq P_{\mathbf{a}}$ by Lemma 3.3. It is not difficult to see that $R_k = C((1 - \beta^k)\mathbf{a}, (1 + \Delta)\beta^k) \supseteq C((1 - \beta^k)\mathbf{a}, \beta^k) = Q_k$. Next, we show that $R_k = C((1 - \beta^k)\mathbf{a}, (1 + \Delta)\beta^k) \supseteq C((1 - \beta^{k+1})\mathbf{a}, \beta^{k+1}) = Q_{k+1}$. Let $\mathbf{x} \in Q_{k+1}$, then

$$\begin{aligned}\|\mathbf{x} - (1 - \beta^{k+1})\mathbf{a}\|_1 &= \|\mathbf{x} - (1 - \beta^k)\mathbf{a} - (\beta^k - \beta^{k+1})\mathbf{a}\|_1 \\ &\geq \|\mathbf{x} - (1 - \beta^k)\mathbf{a}\|_1 - (\beta^k - \beta^{k+1})\|\mathbf{a}\|_1\end{aligned}$$

and it implies that

$$\begin{aligned}\|\mathbf{x} - (1 - \beta^k)\mathbf{a}\|_1 &\leq \beta^{k+1} + (\beta^k - \beta^{k+1})\|\mathbf{a}\|_1 \\ &= \beta^{k+1} + \beta^k(1 - \beta)\|\mathbf{a}\|_1 \\ &= \beta^{k+1} + \beta^k\Delta \\ &\leq (1 + \Delta)\beta^k.\end{aligned}$$

Thus $R_k \supseteq Q_{k+1}$. Clearly a cross-polytope R_k is convex, we obtain the claim $R_k \supseteq \text{conv}(Q_k \cup Q_{k+1})$, which implies that $\text{Vol}(\bigcup_{k=0}^{\infty} R_k) \geq \text{Vol}(P_{\mathbf{a}})$ as we prescribed.

Then, we bound the ratio $\text{Vol}(\bigcup_{k=0}^{\infty} Q_k)/\text{Vol}(P_{\mathbf{a}})$ by $\text{Vol}(\bigcup_{k=0}^{\infty} Q_k)/\text{Vol}(\bigcup_{k=0}^{\infty} R_k)$. For convenience, let

$$\hat{R}_k \stackrel{\text{def}}{=} C((1 + \Delta)(1 - \beta^k)\mathbf{a}, (1 + \Delta)\beta^k).$$

Clearly, $\text{Vol}(\hat{R}_k) = \text{Vol}(R_k)$. It is not difficult to observe that $\text{Vol}(\hat{R}_k \cap \hat{R}_{k+1}) \leq \text{Vol}(R_k \cap R_{k+1})$, which implies that $\text{Vol}(\bigcup_{k=0}^{\infty} \hat{R}_k) \geq \text{Vol}(\bigcup_{k=0}^{\infty} R_k)$. Furthermore,

$$\text{Vol}\left(\bigcup_{k=0}^{\infty} \hat{R}_k\right) = (1 + \Delta)\text{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right)\tag{10}$$

holds since $\bigcup_{k=0}^{\infty} \hat{R}_k$ and $\bigcup_{k=0}^{\infty} Q_k$ are similar. Consequently,

$$\begin{aligned}\frac{\text{Vol}(\bigcup_{k=0}^{\infty} Q_k)}{\text{Vol}(P_{\mathbf{a}})} &\geq \frac{\text{Vol}(\bigcup_{k=0}^{\infty} Q_k)}{\text{Vol}(\bigcup_{k=0}^{\infty} R_k)} \geq \frac{\text{Vol}(\bigcup_{k=0}^{\infty} Q_k)}{\text{Vol}(\bigcup_{k=0}^{\infty} \hat{R}_k)} \\ &= \frac{1}{(1 + \Delta)^n} && \text{(by (10))} \\ &= \frac{1}{(1 + (1 - \beta)\|\mathbf{a}\|_1)^n} && \text{(by (9))} \\ &\geq \frac{1}{\left(1 + \frac{c_1\epsilon}{n\|\mathbf{a}\|_1}\|\mathbf{a}\|_1\right)^n} && \left(\text{since } 1 - \beta \leq \frac{c_1\epsilon}{n\|\mathbf{a}\|_1} \text{ (hypo.)}\right)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\left(1 + \frac{\epsilon}{n}\right)^n} \\
&\geq \left(1 - \frac{\epsilon}{n}\right)^n \\
&\geq 1 - \epsilon.
\end{aligned}$$

We obtain the claim. □

Lemma 3.1 follows Lemmas 3.2 and 3.4.

3.2 Reduction to the intersection of two cross-polytopes

3.2.1 The volume of $\bigcup_{k=0}^{\infty} Q_k$

Section 3.2.1 claims the following.

Lemma 3.5.

$$\text{Vol} \left(\bigcup_{k=0}^{\infty} Q_k \right) = \frac{1}{1 - \beta^n} \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0) \right).$$

The first step of the proof is the following recursive formula.

Lemma 3.6.

$$\bigcup_{k=0}^m Q_k = \left(\bigcup_{k=0}^{m-1} Q_k \setminus Q_{k+1} \right) \dot{\cup} Q_m$$

where $A \dot{\cup} B$ denotes the disjoint union of A and B , meaning that $A \dot{\cup} B = A \cup B$ and $A \cap B = \emptyset$.

Lemma 3.6 is seemingly trivial, where the point is the following claim.

Claim 1.

$$\left(\bigcup_{k=0}^m Q_k \right) \cap Q_{m+1} = Q_m \cap Q_{m+1}.$$

Proof of Claim 1. The inclusion “ \supseteq ” is clear. We prove the other inclusion “ \subseteq .” Suppose for an arbitrary $k \in \{0, 1, \dots, m-1\}$ that $\mathbf{x} \in Q_k \cap Q_{m+1}$ holds. Then $\mathbf{x} \in Q_k$ implies $\langle \mathbf{x} - (1 - \beta^k)\mathbf{a}, \boldsymbol{\sigma} \rangle = \langle \mathbf{x} - \mathbf{a}, \boldsymbol{\sigma} \rangle + \langle \beta^k \mathbf{a}, \boldsymbol{\sigma} \rangle \leq \beta^k$ holds for any $\boldsymbol{\sigma} \in \{-1, 1\}^n$, and $\mathbf{x} \in Q_{m+1}$ implies $\langle \mathbf{x} - (1 - \beta^{m+1})\mathbf{a}, \boldsymbol{\sigma} \rangle = \langle \mathbf{x} - \mathbf{a}, \boldsymbol{\sigma} \rangle + \langle \beta^{m+1} \mathbf{a}, \boldsymbol{\sigma} \rangle \leq \beta^{m+1}$ holds for any $\boldsymbol{\sigma} \in \{-1, 1\}^n$. This means that if $\mathbf{x} \in Q_k \cap Q_{m+1}$ then $\langle \mathbf{x} - \mathbf{a}, \boldsymbol{\sigma} \rangle \leq \min\{\beta^k(1 - \langle \mathbf{a}, \boldsymbol{\sigma} \rangle), \beta^{m+1}(1 - \langle \mathbf{a}, \boldsymbol{\sigma} \rangle)\}$. It is not difficult to see that $\min\{\beta^k(1 - \langle \mathbf{a}, \boldsymbol{\sigma} \rangle), \beta^{m+1}(1 - \langle \mathbf{a}, \boldsymbol{\sigma} \rangle)\} \leq \beta^m(1 - \langle \mathbf{a}, \boldsymbol{\sigma} \rangle)$. Thus we obtain the claim. □

Proof of Lemma 3.6. The claim is trivial when $m = 1$. Inductively assuming that the claim holds in case

of m , we obtain that

$$\begin{aligned}
\bigcup_{k=0}^{m+1} Q_k &= \bigcup_{k=0}^m Q_k \cup Q_{m+1} \\
&= \left(\left(\dot{\bigcup}_{k=0}^{m-1} Q_k \setminus Q_{k+1} \right) \dot{\cup} Q_m \right) \cup Q_{m+1} && \text{(Induction hypo.)} \\
&= \left(\dot{\bigcup}_{k=0}^{m-1} Q_k \setminus Q_{k+1} \right) \dot{\cup} (Q_m \cup Q_{m+1}) && \text{(by Claim 1)} \\
&= \left(\dot{\bigcup}_{k=0}^{m-1} Q_k \setminus Q_{k+1} \right) \dot{\cup} ((Q_m \setminus Q_{m+1}) \dot{\cup} Q_{m+1}) \\
&= \left(\dot{\bigcup}_{k=0}^m Q_k \setminus Q_{k+1} \right) \dot{\cup} Q_m
\end{aligned}$$

which is the claim in case of $m + 1$. \square

The second step of the proof of Lemma 3.7 is the following lemma.

Lemma 3.7.

$$\text{Vol}(Q_k \setminus Q_{k+1}) = \beta^{nk} \text{Vol}(Q_0 \setminus Q_1).$$

Proof. It is easy to see that $\text{Vol}(Q_k)/\text{Vol}(Q_0) = \text{Vol}(C((1 - \beta^k)\mathbf{a}, \beta^k))/\text{Vol}(C(\mathbf{0}, 1)) = \beta^{nk}$ holds, $\text{Vol}(Q_{k+1})/\text{Vol}(Q_1) = \beta^{nk}$ as well. Using the bijective map $\eta_k: Q_k \rightarrow Q_0$ defined by (8) in Lemma 3.2, it is also not difficult to see that $\text{Vol}(Q_k \cap Q_{k+1})/\text{Vol}(Q_0 \cap Q_1) = \beta^{nk}$. Considering the inclusion-exclusion, we obtain the claim. \square

Now, we are ready to prove Lemma 3.5.

Proof of Lemma 3.5.

$$\begin{aligned}
\text{Vol} \left(\bigcup_{k=0}^{\infty} Q_k \right) &= \text{Vol} \left(\left(\dot{\bigcup}_{k=0}^{\infty} Q_k \setminus Q_{k+1} \right) \dot{\cup} Q_{\infty} \right) && \text{(by Lemma 3.6)} \\
&= \sum_{k=0}^{\infty} \text{Vol}(Q_k \setminus Q_{k+1}) + \text{Vol}(Q_{\infty}) \\
&= \sum_{k=0}^{\infty} \beta^{nk} \text{Vol}(Q_0 \setminus Q_1) && \text{(by Lemma 3.7)} \\
&= \frac{1}{1 - \beta^n} \text{Vol}(Q_0 \setminus Q_1) \\
&= \frac{1}{1 - \beta^n} \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0) \right)
\end{aligned}$$

\square

3.2.2 The volume of $\bigcup_{k=0}^{\infty} Q_k$

A reader who are familiar with approximation may worry about the subtraction $\frac{2^n}{n!} - \text{Vol}(Q_0 \cap Q_1)$ in Lemma 3.5. Section 3.2.2 claims the following.

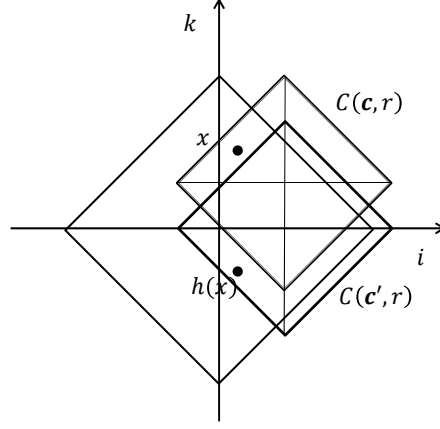


Figure 2: \mathbf{x} and $h(\mathbf{x})$.

Lemma 3.8. When $1 - \beta \geq \frac{c_2 \epsilon}{n \|\mathbf{a}\|_1}$ where $0 < c_2 \epsilon < 1$,

$$\text{Vol}(Q_0 \cap Q_1) \leq \frac{1}{1 + \frac{c_2 \epsilon}{2n}} \frac{2^n}{n!}.$$

Intuitively, Lemma 3.8 implies that $\frac{2^n}{n!} - \text{Vol}(Q_0 \cap Q_1)$ is large enough, and an approximation of $\text{Vol}(Q_0 \cap Q_1)$ provides a good approximation of $\text{Vol}(\bigcup_{k=0}^{\infty} Q_k)$, and hence $\text{Vol}(P_{\mathbf{a}})$. A detailed argument on our FPTAS of $\text{Vol}(P_{\mathbf{a}})$ will be described in Section 3.3.

As a preliminary of a proof of Lemma 3.8, we give Lemmas 3.9 and 3.11.

Lemma 3.9. Let $\mathbf{c} \in \mathbb{R}_{\geq 0}^n$ and $\mathbf{c}' \in \mathbb{R}_{\geq 0}^n$ be given by

$$\begin{aligned} \mathbf{c} &= (c_1, c_2, \dots, c_{k-1}, c_k, 0, \dots, 0) \\ \mathbf{c}' &= (c_1, c_2, \dots, c_{k-1}, 0, 0, \dots, 0) \end{aligned}$$

for some $k \in \{2, 3, \dots, n\}$, i.e., \mathbf{c}' is given by replacing the k -th component of \mathbf{c} by 0. Then,

$$\text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r)) \leq \text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}', r)). \quad (11)$$

Proof. For any $\mathbf{x} = (x_1, \dots, x_n) \in C(\mathbf{c}, r)$, we define a map $h: C(\mathbf{c}, r) \rightarrow \mathbb{R}^n$ such that $\mathbf{x}' = h(\mathbf{x})$ satisfies

$$x'_i = \begin{cases} c_i - x_i & (\text{for } i = k) \\ x_i & (\text{otherwise}) \end{cases}$$

(see Figure 2). Notice that h is a map from $C(\mathbf{c}, r)$ to $C(\mathbf{c}', r)$ in fact, and it is bijective and measure preserving. Now, suppose that $\mathbf{x} \in C(\mathbf{c}, r)$ satisfies both $\mathbf{x} \in C(\mathbf{0}, 1)$ and $\mathbf{x} \notin C(\mathbf{c}', r)$, i.e.,

$$\sum_{i=1}^k |x_i - c_i| + \sum_{i=k+1}^n |x_i| \leq r, \quad (12)$$

$$\sum_{i=1}^n |x_i| \leq 1, \quad \text{and} \quad (13)$$

$$\sum_{i=1}^{k-1} |x_i - c_i| + \sum_{i=k}^n |x_i| > r \quad (14)$$

hold. Then, we claim that $\mathbf{x}' = h(\mathbf{x}) \in C(\mathbf{0}, 1)$ and $\mathbf{x}' \notin C(\mathbf{c}, r)$. This implies (11) since h is measure preserving. Now we show the claim. For convenience let

$$D := \sum_{i=1}^{k-1} |x_i - c_i| + \sum_{i=k+1}^n |x_i|$$

then (12) implies $D + |x_k - c_k| \leq r$ and (14) implies $D + |x_k| > r$. As a consequence, we obtain that

$$|x_k - c_k| < |x_k|. \quad (15)$$

We also remark that $|x'_k| = |x_k - c_k|$ by the definition of h . Then

$$\begin{aligned} \|\mathbf{x}'\|_1 &= \sum_{i=1}^n |x'_i| = (\sum_{i=1}^n |x_i|) - |x_k| + |x_k - c_k| \\ &< \sum_{i=1}^n |x_i| && \text{(by (15))} \\ &\leq 1 && \text{(by (13))} \end{aligned}$$

and $\mathbf{x}' \in C(\mathbf{0}, 1)$. Similarly,

$$\begin{aligned} \|\mathbf{x}' - \mathbf{c}\|_1 &= \sum_{i=1}^{k-1} |x'_i - c_i| + |x'_k| + \sum_{i=k+1}^n |x'_i| \\ &= \sum_{i=1}^{k-1} |x_i - c_i| + |x_k - c_k| + \sum_{i=k+1}^n |x_i| \\ &> \sum_{i=1}^{k-1} |x_i - c_i| + |x_k| + \sum_{i=k+1}^n |x_i| && \text{(by (15))} \\ &\geq r && \text{(by (14))} \end{aligned}$$

and $\mathbf{x}' \notin C(\mathbf{c}, r)$. We obtain the claim. \square

We remark that the volume of the intersection is not monotone decreasing with respect to the L_1 distance between centers, in general. Iteratively applying Lemma 3.9, we see the following.

Corollary 3.10. *Let $\mathbf{c} \in \mathbb{R}_{\geq 0}^n$ and $\mathbf{c}' \in \mathbb{R}_{\geq 0}^n$ be given by*

$$\begin{aligned} \mathbf{c} &= (c_1, c_2, \dots, c_n) \quad \text{and} \\ \mathbf{c}' &= (c_1, 0, \dots, 0), \end{aligned}$$

i.e., \mathbf{c}' is given by replacing each component, except for the first component, of \mathbf{c} by 0. Then,

$$\text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r)) \leq \text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}', r)).$$

Next, we show the following.

Lemma 3.11. *Suppose that r and c satisfies $0 < r < 1$ and $0 < c < 1 + r$. Then,*

$$C(\mathbf{0}, 1) \cap C((c, 0, \dots, 0), r) \subseteq C\left(\left(\frac{1 + (c - r)}{2}, 0, \dots, 0\right), \frac{1 - (c - r)}{2}\right) \quad (16)$$

holds.

Remark that (16) in fact holds by equality, but we here prove only \subseteq , which we will use.

Proof. Suppose that $\mathbf{x} \in C(\mathbf{0}, 1) \cap C((c, 0, \dots, 0), r)$, i.e.,

$$\sum_{i=1}^n |x_i| \leq 1 \quad \text{and} \quad (17)$$

$$|x_1 - c| + \sum_{i=2}^n |x_i| \leq r \quad (18)$$

holds. We consider two cases.

(i) In case that $x_1 \geq \frac{1+(c-r)}{2}$,

$$\begin{aligned}
\left| x_1 - \frac{1+(c-r)}{2} \right| + \sum_{i=2}^n |x_i| &= x_1 - \frac{1+(c-r)}{2} + \sum_{i=2}^n |x_i| \\
&= \sum_{i=1}^n |x_i| - \frac{1+(c-r)}{2} \\
&\leq 1 - \frac{1+(c-r)}{2} \quad (\text{by (17)}) \\
&= \frac{1-(c-r)}{2}
\end{aligned}$$

and we see that $\mathbf{x} \in C\left(\left(\frac{1+(c-r)}{2}, 0, \dots, 0\right), \frac{1-(c-r)}{2}\right)$.

(ii) In case that $x_1 < \frac{1+(c-r)}{2}$,

$$\begin{aligned}
\left| x_1 - \frac{1+(c-r)}{2} \right| + \sum_{i=2}^n |x_i| &= \frac{1+(c-r)}{2} - x_1 + \sum_{i=2}^n |x_i| \\
&= \frac{1+(c-r)}{2} - c + (c - x_1) + \sum_{i=2}^n |x_i| \\
&\leq \frac{1+(c-r)}{2} - c + |x_1 - c| + \sum_{i=2}^n |x_i| \\
&\leq \frac{1+(c-r)}{2} - c + r \quad (\text{by (18)}) \\
&= \frac{1-(c-r)}{2}
\end{aligned}$$

and we see that $\mathbf{x} \in C\left(\left(\frac{1+(c-r)}{2}, 0, \dots, 0\right), \frac{1-(c-r)}{2}\right)$. □

Now we prove Lemma 3.8.

Proof of Lemma 3.8. Recall that $Q_1 = C((1-\beta)\mathbf{a}, \beta)$. By Corollary 3.10 and Lemma 3.11,

$$\begin{aligned}
\text{Vol}(Q_0 \cap Q_1) &\leq \text{Vol}(C(\mathbf{0}, 1) \cap C(((1-\beta)a_1, 0, \dots, 0), \beta)) \quad (\text{by Corollary 3.10}) \\
&\leq \text{Vol}\left(C\left(\left(\frac{1+((1-\beta)a_1 - \beta)}{2}, 0, \dots, 0\right), \frac{1-((1-\beta)a_1 - \beta)}{2}\right)\right) \\
&\quad (\text{by Lemma 3.11}) \\
&= \left(\frac{1-((1-\beta)a_1 - \beta)}{2}\right)^n \text{Vol}(C(\mathbf{0}, 1)) \\
&= \left(\frac{2-(1-\beta)(a_1+1)}{2}\right)^n \text{Vol}(C(\mathbf{0}, 1)) \\
&= \left(1-(1-\beta)\frac{a_1+1}{2}\right)^n \text{Vol}(C(\mathbf{0}, 1)) \\
&\leq \left(1-\frac{c_2\epsilon}{n\|\mathbf{a}\|_1}\frac{a_1+1}{2}\right)^n \text{Vol}(C(\mathbf{0}, 1)) \quad \left(\text{since } 1-\beta \geq \frac{c_2\epsilon}{n\|\mathbf{a}\|_1} \text{ (hypo.)}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \left(1 - \frac{c_2\epsilon}{2n^2}\right)^n \text{Vol}(C(\mathbf{0}, 1)) && \left(\text{since } \frac{a_1 + 1}{2\|\mathbf{a}\|_1} \geq \frac{a_1 + 1}{2na_1} \geq \frac{1}{2n}\right) \\
&\leq \frac{1}{\left(1 + \frac{c_2\epsilon}{2n^2}\right)^n} \text{Vol}(C(\mathbf{0}, 1)) \\
&\leq \frac{1}{1 + \frac{c_2\epsilon}{2n}} \text{Vol}(C(\mathbf{0}, 1))
\end{aligned}$$

and we obtain the claim. \square

3.3 Approximation algorithm and analysis

Based on Lemma 3.1 in Section 3.1 and Lemma 3.5 in Section 3.2, we give an FPTAS for $\text{Vol}(P_{\mathbf{a}})$ where we assume an algorithm to approximate $\text{Vol}(Q_0 \cap Q_1)$. For convenience of arguments, we assume $0 < \epsilon \leq 1/2$, but it is clearly not essential⁸.

Algorithm 1 ($(1 \pm \epsilon)$ -approximation ($0 < \epsilon \leq 1/2$)).

Input: $\mathbf{a} \in \mathbb{Z}_+^n$;

1. Set parameter $\beta := 1 - \frac{\epsilon}{2n\|\mathbf{a}\|_1}$; 2. Approximate $I \stackrel{\text{def}}{=} \text{Vol}(C(\mathbf{0}, 1) \cap C((1 - \beta)\mathbf{a}, \beta))$ by Z such that

$$I \leq Z \leq \left(1 + \frac{\epsilon^2}{4n}\right) I;$$

3. Output

$$\widehat{V} = \frac{1 + \epsilon}{1 - \beta^n} \left(\frac{2^n}{n!} - Z \right).$$

Lemma 3.12. *The output \widehat{V} of Algorithm 1 satisfies*

$$(1 - \epsilon) \text{Vol}(P_{\mathbf{a}}) \leq \widehat{V} \leq (1 + \epsilon) \text{Vol}(P_{\mathbf{a}}).$$

Before proving Lemma 3.12, we check the time complexity of Algorithm 1. In Section 4, we will give an FPTAS for $\text{Vol}(Q_0 \cap Q_1)$. Theorem 4.1 appearing there implies that the time complexity of Step 2 of Algorithm 1 is $O(n^7(n/\epsilon^2)^3) = O(n^{10}\epsilon^{-6})$. Thus, we obtain Theorem 1.1 by Lemma 3.12.

As a preliminary of Lemma 3.12, we show the following.

Lemma 3.13. *Suppose that $1 - \beta \geq \frac{c_2\epsilon}{n\|\mathbf{a}\|_1}$ holds where $0 < c_2\epsilon < 1$. If we have an approximation Z of $\text{Vol}(Q_1 \cap Q_0)$ satisfying*

$$\text{Vol}(Q_0 \cap Q_1) \leq Z \leq \left(1 + \frac{c_2\epsilon^2}{2n}\right) \text{Vol}(Q_0 \cap Q_1) \quad (19)$$

then $\text{Vol}(Q_0) - Z = \frac{2^n}{n!} - Z$ satisfies that

$$(1 - \epsilon) \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0) \right) \leq \left(\frac{2^n}{n!} - Z \right) \leq \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0) \right). \quad (20)$$

⁸ For $\epsilon > 1/2$, use Algorithm 1 with $\epsilon = 1/2$.

Proof. The second inequality of (20) is easy from the assumption (19), such that

$$\frac{2^n}{n!} - Z \leq \frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0)$$

holds. For the first inequality of (20), (19)

$$\begin{aligned} \frac{2^n}{n!} - Z &\geq \frac{2^n}{n!} - \left(1 + \frac{c_2 \epsilon^2}{2n}\right) \text{Vol}(Q_1 \cap Q_0) && \text{(by (19))} \\ &= \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0)\right) - \frac{c_2 \epsilon^2}{2n} \text{Vol}(Q_1 \cap Q_0) \\ &= \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0)\right) \left(1 - \frac{c_2 \epsilon^2}{2n} \frac{\text{Vol}(Q_1 \cap Q_0)}{\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0)}\right) \end{aligned} \quad (21)$$

holds. Since the hypothesis $1 - \beta \geq \frac{c_2 \epsilon}{n \|\mathbf{a}\|_1}$, Lemma 3.8 implies that

$$\text{Vol}(Q_0 \cap Q_1) \leq \frac{1}{1 + \frac{c_2 \epsilon}{2n}} \frac{2^n}{n!}$$

and hence

$$\begin{aligned} \frac{\text{Vol}(Q_0 \cap Q_1)}{\frac{2^n}{n!} - \text{Vol}(Q_0 \cap Q_1)} &= \frac{1}{\frac{\frac{2^n}{n!}}{\text{Vol}(Q_0 \cap Q_1)} - 1} \\ &\leq \frac{1}{1 + \frac{c_2 \epsilon}{2n} - 1} \\ &= \frac{2n}{c_2 \epsilon} \end{aligned}$$

holds. Thus,

$$\begin{aligned} (21) &\geq \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0)\right) \left(1 - \frac{c_2 \epsilon^2}{2n} \frac{2n}{c_2 \epsilon}\right) \\ &= \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0)\right) (1 - \epsilon) \end{aligned}$$

holds, and we obtain the claim. \square

Corollary 3.14. *Let $1 - \beta = \frac{c\epsilon}{n \|\mathbf{a}\|_1}$ with $\frac{1}{4} \leq c \leq \frac{1}{2}$. If we have an approximation Z of $\text{Vol}(Q_1 \cap Q_0)$ satisfying*

$$\text{Vol}(Q_0 \cap Q_1) \leq Z \leq \left(1 + \frac{c\epsilon^2}{2n}\right) \text{Vol}(Q_0 \cap Q_1) \quad (22)$$

then

$$\widehat{V} := \frac{1 + \epsilon}{1 - \beta^n} \left(\frac{2^n}{n!} - Z\right)$$

satisfies

$$(1 - \epsilon) \text{Vol}(P\mathbf{a}) \leq \widehat{V} \leq (1 + \epsilon) \text{Vol}(P\mathbf{a}).$$

Proof. Recall Lemma 3.5, that is

$$\text{Vol} \left(\bigcup_{k=0}^{\infty} Q_k \right) = \frac{1}{1 - \beta^n} \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0) \right).$$

By Lemma 3.1, the hypothesis $1 - \beta \leq \frac{\frac{1}{2}\epsilon}{n\|\mathbf{a}\|_1}$ implies that

$$\left(1 - \frac{1}{2}\epsilon \right) \text{Vol}(P_{\mathbf{a}}) \leq \text{Vol} \left(\bigcup_{k=0}^{\infty} Q_k \right) \leq \text{Vol}(P_{\mathbf{a}})$$

holds. Thus, (22) implies that

$$\begin{aligned} \widehat{V} &= \frac{1 + \epsilon}{1 - \beta^n} \left(\frac{2^n}{n!} - Z \right) \\ &\leq \frac{1 + \epsilon}{1 - \beta^n} \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0) \right) \\ &\leq (1 + \epsilon) \text{Vol}(P_{\mathbf{a}}) \end{aligned}$$

and we obtain the upper bound. Similarly,

$$\begin{aligned} \widehat{V} &= \frac{1 + \epsilon}{1 - \beta^n} \left(\frac{2^n}{n!} - Z \right) \\ &\geq \frac{1 + \epsilon}{1 - \beta^n} (1 - \epsilon) \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0) \right) \\ &\geq (1 - \epsilon^2) \frac{1}{1 - \beta^n} \left(\frac{2^n}{n!} - \text{Vol}(Q_1 \cap Q_0) \right) \\ &\geq (1 - \epsilon^2) \left(1 - \frac{\epsilon}{2} \right) \text{Vol}(P_{\mathbf{a}}) \\ &\geq (1 - \epsilon) \text{Vol}(P_{\mathbf{a}}) \quad \left(\text{by assumption } \epsilon \leq \frac{1}{2} \right) \end{aligned}$$

and we obtain the claim. \square

Now, Lemma 3.12 is immediate from Corollary 3.14.

4 The Volume of the Intersection of Two Cross-polytopes

This section gives an FPTAS for the volume of the intersection of two cross-polytopes in the n -dimensional space. Without loss of generality⁹, we are concerned with $\text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r))$ for $\mathbf{c} \geq \mathbf{0}$ and r ($0 < r \leq 1$). This section establishes the following.

Theorem 4.1. *For any δ ($0 < \delta < 1$), there exists a deterministic algorithm which outputs a value Z satisfying $\text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r)) \leq Z \leq (1 + \delta) \text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r))$ for any input $\mathbf{c} \geq \mathbf{0}$ and r ($0 < r \leq 1$) satisfying $\|\mathbf{c}\|_1 \leq r$, and runs in $O(n^7 \delta^{-3})$ time.*

The assumption that $\|\mathbf{c}\|_1 \leq r$ implies both centers $\mathbf{0}$ and \mathbf{c} are contained in the intersection $C(\mathbf{0}, 1) \cap C(\mathbf{c}, r)$. Note that the assumption does not harm to our main goal Theorem 1.1 (recall Algorithm 1 in Section 3.3). We show in Section 5 that $\text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r))$ remains $\#P$ -hard even on the assumption. We will use the assumption in the proof of Lemma 4.9.

⁹ Remark that $\text{Vol}(C(\mathbf{c}, r) \cap C(\mathbf{c}', r')) = r^n \text{Vol} \left(C(\mathbf{0}, 1) \cap C \left(\frac{(\mathbf{c} - \mathbf{c}')^+}{r}, \frac{r'}{r} \right) \right)$ holds for any $\mathbf{c}, \mathbf{c}' \in \mathbb{R}^n$ and $r, r' \in \mathbb{R}_{>0}$, where $(\mathbf{c} - \mathbf{c}')^+ = (|c_1 - c'_1|, |c_2 - c'_2|, \dots, |c_n - c'_n|)$.

4.1 Preliminary: convolution for the volume

As a preliminary step, Section 4.1 gives a convolution which provides $\text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r))$. Let $\Psi_0: \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $\Psi_0(u, v) = 1$ if $u \geq 0$ and $v \geq 0$, otherwise $\Psi_0(u, v) = 0$. Inductively, we define $\Psi_i: \mathbb{R}^2 \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, n$ by

$$\Psi_i(u, v) \stackrel{\text{def}}{=} \int_{-1}^1 \Psi_{i-1}(u - |s|, v - |s - c_i|) ds \quad (23)$$

for $u, v \in \mathbb{R}$. We remark that $\Psi_i(u, v) = 0$ holds if $u \leq 0$ or $v \leq 0$, for any $i = 1, 2, \dots, n$ by the definition.

Lemma 4.2.

$$\Psi_n(1, r) = \text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r)).$$

To prove Lemma 4.2, it might be helpful to introduce a probability space. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a uniform random variable over $[-1, 1]^n$, i.e., X_i ($i = 1, \dots, n$) are (mutually) independent. Then,

$$\Pr[X \in C(\mathbf{0}, 1) \cap C(\mathbf{c}, r)] = \frac{\text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r))}{\text{Vol}([-1, 1]^n)} = \frac{1}{2^n} \text{Vol}(C(\mathbf{0}, 1) \cap C(\mathbf{c}, r)) \quad (24)$$

holds.

Lemma 4.3. For any $i = 1, 2, \dots, n$,

$$\frac{1}{2^i} \Psi_i(u, v) = \Pr \left[\left(\sum_{j=1}^i |X_j| \leq u \right) \wedge \left(\sum_{j=1}^i |X_j - c_j| \leq v \right) \right]$$

for any $u, v \in \mathbb{R}$.

Proof. First, we prove the claim for $i = 1$. Considering that $\Psi_0(u, v)$ is an indicator function,

$$\begin{aligned} \Psi_1(u, v) &= \int_{-1}^1 \Psi_0(u - |s|, v - |s - c_1|) ds \\ &= |\{s \in [-1, 1] \mid (u - |s| \geq 0) \wedge (v - |s - c_1| \geq 0)\}| \\ &= 2 \frac{|\{s \in [-1, 1] \mid (u - |s| \geq 0) \wedge (v - |s - c_1| \geq 0)\}|}{|[-1, 1]|} \\ &= 2 \Pr[(0 \leq u - |X_1|) \wedge (0 \leq v - |X_1 - c_1|)] \\ &= 2 \Pr[(|X_1| \leq u) \wedge (|X_1 - c_1| \leq v)] \end{aligned}$$

and we obtain the claim in the case.

Inductively assuming that the claim for i , we show that the claim for $i + 1$. Let f denote the uniform

density over $[-1, 1]$. Then,

$$\begin{aligned}
& \Pr \left[\left(\sum_{j=1}^{i+1} |X_j| \leq u \right) \wedge \left(\sum_{j=1}^{i+1} |X_j - c_j| \leq v \right) \right] \\
&= \int_{-\infty}^{\infty} \Pr \left[\left(\sum_{j=1}^{i+1} |X_j| \leq u \right) \wedge \left(\sum_{j=1}^{i+1} |X_j - c_j| \leq v \right) \mid X_{i+1} = s \right] f(s) ds \\
&= \int_{-1}^1 \Pr \left[\left(\sum_{j=1}^{i+1} |X_j| \leq u \right) \wedge \left(\sum_{j=1}^{i+1} |X_j - c_j| \leq v \right) \mid X_{i+1} = s \right] \frac{1}{2} ds \\
&= \frac{1}{2} \int_{-1}^1 \Pr \left[\left(\sum_{j=1}^i |X_j| + |s| \leq u \right) \wedge \left(\sum_{j=1}^i |X_j - c_j| + |s - c_{i+1}| \leq v \right) \right] ds \\
&= \frac{1}{2} \int_{-1}^1 \Pr \left[\left(\sum_{j=1}^i |X_j| \leq u - |s| \right) \wedge \left(\sum_{j=1}^i |X_j - c_j| \leq v - |s - c_{i+1}| \right) \right] ds \\
&= \frac{1}{2} \int_{-1}^1 \frac{1}{2^i} \Psi_i(u - |s|, v - |s - c_{i+1}|) ds \\
&= \frac{1}{2^{i+1}} \Psi_{i+1}(u, v)
\end{aligned}$$

and we obtain the claim. \square

Now, Lemma 4.2 is easy from Lemma 4.3 and (24).

4.2 Idea for approximation

Our FPTAS is based on an approximation of $\Psi_i(u, v)$. Let $G_0(u, v) = \Psi_0(u, v)$ for any $u, v \in \mathbb{R}$, i.e., $G_0(u, v) = 1$ if $u \geq 0$ and $v \geq 0$, otherwise $G_0(u, v) = 0$. Inductively assuming $G_{i-1}(u, v)$, we define

$$\overline{G}_i(u, v) \stackrel{\text{def}}{=} \int_{-1}^1 G_{i-1}(u - |s|, v - |s - c_i|) ds \quad (25)$$

for $u, v \in \mathbb{R}$, for convenience. Then, let $G_i(u, v)$ be a staircase approximation of $\overline{G}_i(u, v)$, given by

$$G_i(u, v) \stackrel{\text{def}}{=} \begin{cases} \overline{G}_i\left(\frac{1}{M}k, \frac{r}{M}\ell\right) & \left(\begin{array}{l} \text{if } \frac{1}{M}(k-1) < u \leq \frac{1}{M}k \quad (k = 1, 2, \dots), \text{ and} \\ \frac{r}{M}(\ell-1) < v \leq \frac{r}{M}\ell \quad (\ell = 1, 2, \dots). \end{array} \right) \\ 0 & \text{(otherwise)} \end{cases} \quad (26)$$

for any $u, v \in \mathbb{R}$. Thus, we remark that

$$G_i(u, v) = G_i\left(\frac{1}{M}\lceil Mu \rceil, \frac{r}{M}\lceil \frac{M}{r}v \rceil\right) \quad (27)$$

holds for any $u, v \in \mathbb{R}$, by the definition. Section 4.3 will show that $G_i(u, v)$ approximates $\Psi_i(u, v)$ well.

In the rest of Section 4.2, we briefly comment on the computation of G_i . First, remark that (25) implies that $\overline{G}_i(u, v)$ is computed only from $G_{i-1}(u', v')$ for $u' \leq u$ and $v' \leq v$, i.e., we do not need to know $G_{i-1}(u', v')$ for $u' > u$ or $v' > v$. Second, remark (27) implies that $G_i(u, v)$ for $u \leq 1$ and $v \leq r$ takes (at most) $(M+1)^2$ different values. Precisely, let

$$\Gamma \stackrel{\text{def}}{=} \left\{ \frac{1}{M}(k, r\ell) \mid k = 0, 1, 2, \dots, M, \ell = 0, 1, 2, \dots, M \right\}$$

then $G_i(u, v)$ for $(u, v) \in \Gamma$ provides all possible values of $G_i(u, v)$ for $u \leq 1$ and $v \leq r$, since (27).

Then, we explain how to compute $G_i(u, v)$ for $(u, v) \in \Gamma$ from G_{i-1} . For an arbitrary $(u, v) \in \Gamma$, let

$$\begin{aligned} S(u) &\stackrel{\text{def}}{=} \{s \in [-1, 1] \mid u - |s| = \frac{1}{M}k \ (k = 0, 1, 2, \dots, M)\} \\ &= \{s \in [-1, 1] \mid s = \pm(u - \frac{1}{M}k) \ (k = 0, 1, 2, \dots, M)\}, \end{aligned}$$

let

$$\begin{aligned} S_i(v) &\stackrel{\text{def}}{=} \{s \in [-1, 1] \mid v - |s - c_i| = \frac{r}{M}\ell \ (\ell = 0, 1, 2, \dots, M)\} \\ &= \{s \in [-1, 1] \mid s = c_i \pm (v - \frac{r}{M}\ell) \ (\ell = 0, 1, 2, \dots, M)\}, \end{aligned}$$

and let

$$T_i(u, v) \stackrel{\text{def}}{=} S(u) \cup S_i(v) \cup \{-1, 0, c_i, 1\}$$

Suppose t_0, t_1, \dots, t_m be an ordering of all elements of $T_i(u, v)$ such that $t_i \leq t_{i+1}$ for any $i = 0, 1, \dots, m$, where $m = |T_i(u, v)|$. Then, we can compute $G_i(u, v)$ for any $(u, v) \in \Gamma$ by

$$\begin{aligned} G_i(u, v) &= \overline{G}_i(u, v) \\ &= \int_{-1}^1 G_{i-1}(u - |s|, v - |s - c_i|) ds \\ &= \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} G_{i-1}(u - |s|, v - |s - c_i|) ds \\ &= \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} G_{i-1} \left(\frac{1}{M} \lceil M(u - |t_{j+1}|) \rceil, \frac{r}{M} \lceil \frac{M}{r} (v - |t_{j+1} - c_i|) \rceil \right) ds \quad (\text{by (27)}) \\ &= \sum_{j=0}^{m-1} (t_{j+1} - t_j) G_{i-1} \left(\frac{1}{M} \lceil M(u - |t_{j+1}|) \rceil, \frac{r}{M} \lceil \frac{M}{r} (v - |t_{j+1} - c_i|) \rceil \right) \end{aligned} \quad (28)$$

where we remark again that the terms of (28) consist of $G_{i-1}(u, v)$ for $(u, v) \in \Gamma$.

4.3 Algorithm and analysis

Based on the arguments in Section 4.2, our algorithm is described as follows.

Algorithm 2 (for $(1 + \delta)$ -approximation ($0 < \delta \leq 1$)).

Input: $\mathbf{c} \in \mathbb{Q}_{\geq 0}^n, r \in \mathbb{Q}$ ($0 \leq r \leq 1$);

1. Set $M := \lceil 4n^2\delta^{-1} \rceil$;
2. Set $G_0(u, v) := 1$ for $(u, v) \in \Gamma$, otherwise $G_0(u, v) := 0$;
3. For $i := 1, \dots, n$,
4. For $(u, v) \in \Gamma$,
5. Compute $G_i(u, v)$ from G_{i-1} by (28);
6. Output $G_n(1, r)$.

Lemma 4.4. *The running time of Algorithm 2 is $O(n^7\delta^{-3})$.*

Proof. First, we are concerned with the running time of line 5. The equation (28) is a sum consisting of m terms, where clearly $m \leq 2M + 4 = O(M)$. We specially note that the ordering $t_0, t_1, t_2, \dots, t_m$ of $T_i(u, v)$ is obtained in $O(M)$ time, and hence line 5 runs in $O(M)$ time. Since $|\Gamma| = O(M^2)$, it is easy to see that the running time of Algorithm 2 is $O(nM^3)$. Since $M = O(n^2\delta^{-1})$ by line 2, we obtain the claim. \square

Theorem 4.1 is immediate from Lemma 4.4 and the following Lemma 4.5.

Lemma 4.5.

$$\Psi_n(1, r) \leq G_n(1, r) \leq (1 + \delta)\Psi(1, r).$$

The rest of Section 4.3 proves Lemma 4.5. As a preliminary we remark the following observation from Lemma 4.3.

Observation 4.6. $\Psi_i(u, v)$ is monotone non-decreasing with respect to u , as well as v .

Proof. Suppose that $u \leq u'$ and $v \leq v'$ hold. Lemma 4.3 implies that

$$\begin{aligned} \Psi_i(u, v) &= 2^i \Pr \left[\left(\sum_{j=1}^i |X_j| \leq u \right) \wedge \left(\sum_{j=1}^i |X_j - c_j| \leq v \right) \right] \\ &\leq 2^i \Pr \left[\left(\sum_{j=1}^i |X_j| \leq u' \right) \wedge \left(\sum_{j=1}^i |X_j - c_j| \leq v' \right) \right] = \Psi_i(u', v'). \end{aligned}$$

□

First, we give a lower bound of $G_i(u, v)$.

Lemma 4.7. $\Psi_i(u, v) \leq G_i(u, v)$ for any $u, v \in \mathbb{R}$ and $i = 1, 2, \dots, n$.

Proof. We give an inductive proof. $\Psi_0(u, v) = G_0(u, v)$ by the definition. Inductively assuming the claim for i , we show the claim for $i + 1$ as follows:

$$\begin{aligned} G_{i+1}(u, v) &= \overline{G}_{i+1}\left(\frac{1}{M}\lceil Mu \rceil, \frac{r}{M}\lceil \frac{M}{r}v \rceil\right) && \text{(Recall (26) and (27))} \\ &= \int_{-1}^1 G_i\left(\frac{1}{M}\lceil Mu \rceil - |s|, \frac{r}{M}\lceil \frac{M}{r}v \rceil - |s - c_i|\right) ds \\ &\geq \int_{-1}^1 \Psi_i\left(\frac{1}{M}\lceil Mu \rceil - |s|, \frac{r}{M}\lceil \frac{M}{r}v \rceil - |s - c_i|\right) ds && \text{(Induction hypo.)} \\ &= \Psi_{j+1}\left(\frac{1}{M}\lceil Mu \rceil, \frac{r}{M}\lceil \frac{M}{r}v \rceil\right) \\ &\geq \Psi_{j+1}(u, v) && \text{(By Obs. 4.6)} \end{aligned}$$

and we obtain the claim. □

Next, we give an upper bound of $G_i(u, v)$.

Lemma 4.8. $G_i(u, v) \leq \Psi_i(u + \frac{1}{M}i, v + \frac{r}{M}i)$ for any $u, v \in \mathbb{R}$ and $i = 1, 2, \dots, n$.

Proof. The proof is an induction on n . By the definition that $G_0(u, v) = \Psi_0(u, v)$ for any u, v , the claim is clear when $n = 0$. Inductively assuming the claim holds when $n = i$, meaning that $G_i(u, v) \leq \Psi_i(u + \frac{1}{M}i, v + \frac{r}{M}i)$ holds, we show the claim when $n = i + 1$. By the definition of $G_i(u, v)$ and $\overline{G}_{i+1}(u, v)$, we

have

$$\begin{aligned}
G_{i+1}(u, v) &= \overline{G}_{i+1}(\frac{1}{M}\lceil Mu \rceil, \frac{r}{M}\lceil \frac{M}{r}v \rceil) && \text{(Recall (26) and (27))} \\
&= \int_{-1}^1 G_i(\frac{1}{M}\lceil Mu \rceil - |s|, \frac{r}{M}\lceil \frac{M}{r}v \rceil - |s - c_i|) ds \\
&\leq \int_{-1}^1 \Psi_i(\frac{1}{M}\lceil Mu \rceil - |s| + \frac{1}{M}i, \frac{r}{M}\lceil \frac{M}{r}v \rceil - |s - c_i| + \frac{r}{M}i) ds && \text{(Induction hypo.)} \\
&\leq \int_{-1}^1 \Psi_i(u + \frac{1}{M} - |s| + \frac{1}{M}i, v + \frac{r}{M} - |s - c_i| + \frac{r}{M}i) ds && \left(\begin{array}{l} \text{By Obs. 4.6. Remark} \\ \frac{1}{M}\lceil Mu \rceil \leq u + \frac{1}{M}, \\ \frac{r}{M}\lceil \frac{M}{r}v \rceil \leq v + \frac{r}{M}. \end{array} \right) \\
&= \int_{-1}^1 \Psi_i(u + \frac{1}{M}(i+1) - |s|, v + \frac{r}{M}(i+1) - |s - c_i|) ds \\
&= \Psi_{i+1}(u + \frac{1}{M}(i+1), v + \frac{r}{M}(i+1))
\end{aligned}$$

and we obtain the claim. \square

Lemma 4.9. When $\|c\|_1 \leq r$,

$$\frac{\Psi(1, r)}{\Psi(1 + \frac{n}{M}, r(1 + \frac{n}{M}))} \geq \left(\frac{M}{M+n} \right)^{2n}.$$

Proof. We prove the following two inequalities,

$$\frac{\Psi(1, r)}{\Psi(1 + \frac{n}{M}, r)} \geq \left(\frac{M}{M+n} \right)^n, \quad \text{and} \quad (29)$$

$$\frac{\Psi(1 + \frac{n}{M}, r)}{\Psi(1 + \frac{n}{M}, r(1 + \frac{n}{M}))} \geq \left(\frac{M}{M+n} \right)^n, \quad (30)$$

respectively, where the proofs of (29) and (30) are similar. The claim is clear from (29) and (30).

First we prove (29). For convenience, let

$$K(q) = \{\lambda \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \in C(\mathbf{0}, 1) \cap C(\mathbf{c}, r), \lambda \in \mathbb{R} \text{ such that } 0 \leq \lambda \leq q\}$$

for $q \geq 1$. It is not difficult to see from the definition that

$$\frac{\text{Vol}(K(1))}{\text{Vol}(K(1 + \frac{n}{M}))} = \left(\frac{M}{M+n} \right)^n \quad (31)$$

holds, where we remark that $\text{Vol}(K(1)) = \Psi(1, r)$ since $K(1) = C(\mathbf{0}, 1) \cap C(\mathbf{c}, r)$ by the definition. To claim $\Psi(1 + \frac{n}{M}, r) \leq \text{Vol}(K(1 + \frac{n}{M}))$, we show that

$$C(\mathbf{0}, 1 + \frac{n}{M}) \cap C(\mathbf{c}, r) \subseteq K(1 + \frac{n}{M}) \quad (32)$$

holds. Suppose $\mathbf{y} \in C(\mathbf{0}, 1 + \frac{n}{M}) \cap C(\mathbf{c}, r)$, and we prove $\mathbf{y} \in K(1 + \frac{n}{M})$. More precisely, let $\mathbf{w} = \lambda^{-1}\mathbf{y}$ where $\lambda = 1 + \frac{n}{M}$, and we show $\mathbf{w} \in K(1)$. Since $\mathbf{y} \in C(\mathbf{0}, 1 + \frac{n}{M})$, $\|\mathbf{w}\|_1 = \lambda^{-1}\|\mathbf{y}\|_1 \leq 1$ holds, meaning that $\mathbf{w} \in C(\mathbf{0}, 1)$. Considering $\mathbf{y} \in C(\mathbf{c}, r)$, and the assumption $\|c\|_1 \leq r$, we have

$$\begin{aligned}
\|\mathbf{w} - \mathbf{c}\|_1 &= \|\lambda^{-1}\mathbf{y} - \mathbf{c}\|_1 \\
&= \|\lambda^{-1}(\mathbf{y} - \mathbf{c}) - (1 - \lambda^{-1})\mathbf{c}\|_1 \\
&\leq \lambda^{-1}\|\mathbf{y} - \mathbf{c}\|_1 + (1 - \lambda^{-1})\|\mathbf{c}\|_1 \\
&\leq \lambda^{-1}r + (1 - \lambda^{-1})r \\
&= r
\end{aligned}$$

and hence $\mathbf{w} \in C(\mathbf{c}, r)$. We obtain (32). Carefully recalling Lemma 4.3, $\Psi(1 + \frac{n}{M}, r) = \text{Vol}(C(\mathbf{0}, 1 + \frac{n}{M}) \cap C(\mathbf{c}, r) \cap [-1, 1]^n)$ holds, which implies $\Psi(1 + \frac{n}{M}, r) \leq \text{Vol}(K(1 + \frac{n}{M}))$ with (32). Now, (29) is easy from (31).

The proof of (30) is similar. Let

$$K'(q) = \{ \lambda(\mathbf{x} - \mathbf{c}) \in \mathbb{R}^n \mid \mathbf{x} \in C(\mathbf{0}, 1 + \frac{n}{M}) \cap C(\mathbf{c}, r), \lambda \in \mathbb{R} \text{ such that } 0 \leq \lambda \leq q \}$$

for $q \geq 1$. Then, It is not difficult to see from the definition that

$$\frac{\text{Vol}(K'(1))}{\text{Vol}(K'(1 + \frac{n}{M}))} = \left(\frac{M}{M+n} \right)^n \quad (33)$$

holds, where we remark that $\text{Vol}(K'(1)) = \Psi(1 + \frac{n}{M}, r)$. To claim $\Psi(1 + \frac{n}{M}, r(1 + \frac{n}{M})) \leq \text{Vol}(K'(1 + \frac{n}{M}))$, we show that

$$C(\mathbf{0}, 1 + \frac{n}{M}) \cap C(\mathbf{c}, r(1 + \frac{n}{M})) \subseteq K'(1 + \frac{n}{M}) \quad (34)$$

holds. Suppose $\mathbf{y}' \in C(\mathbf{0}, 1 + \frac{n}{M}) \cap C(\mathbf{c}, r(1 + \frac{n}{M}))$, and we prove $\mathbf{y}' \in K'(1 + \frac{n}{M})$. More precisely, let $\mathbf{w}' = \lambda^{-1}(\mathbf{y}' - \mathbf{c}) + \mathbf{c}$ where $\lambda = 1 + \frac{n}{M}$, and we show $\mathbf{w}' \in K'(1)$. Since $\mathbf{y}' \in C(\mathbf{c}, r(1 + \frac{n}{M}))$, $\|\mathbf{w}' - \mathbf{x}\|_1 = \lambda^{-1}\|\mathbf{y}' - \mathbf{c}\|_1 \leq 1$ holds, meaning that $\mathbf{w}' \in C(\mathbf{0}, 1)$. Considering $\mathbf{y}' \in C(\mathbf{0}, 1 + \frac{n}{M})$, and the assumption $\|\mathbf{c}\|_1 \leq r$, we have

$$\begin{aligned} \|\mathbf{w}'\|_1 &= \|\lambda^{-1}(\mathbf{y}' - \mathbf{c}) + \mathbf{c}\|_1 \\ &= \|\lambda^{-1}\mathbf{y}' + (1 - \lambda^{-1})\mathbf{c}\|_1 \\ &\leq \lambda^{-1}\|\mathbf{y}'\|_1 + (1 - \lambda^{-1})\|\mathbf{c}\|_1 \\ &\leq \lambda^{-1}(1 + \frac{n}{M}) + (1 - \lambda^{-1})r \\ &\leq \lambda^{-1}(1 + \frac{n}{M}) + (1 - \lambda^{-1})(1 + \frac{n}{M}) \quad (\text{since } r \leq 1) \\ &= (1 + \frac{n}{M}) \end{aligned}$$

and hence $\mathbf{w}' \in C(\mathbf{0}, 1 + \frac{n}{M})$. We obtain (34), and hence (30) from (33). Now, we obtain the claim. \square

Now, we prove Lemma 4.5.

Proof of Lemma 4.5. The first inequality is immediate from Lemma 4.7. Then, we show the latter inequality. Lemma 4.9 implies that

$$\begin{aligned} \frac{\Psi_n(1, r)}{\Psi_n(1 + \frac{n}{M}, r(1 + \frac{n}{M}))} &\geq \left(\frac{M}{M+n} \right)^{2n} = \left(\frac{1}{1 + \frac{n}{M}} \right)^{2n} \\ &\geq \left(1 - \frac{n}{M} \right)^{2n} \quad \left(\text{since } (1 + \frac{n}{M})^{2n} (1 - \frac{n}{M})^{2n} \leq 1 \right) \\ &\geq \left(1 - \frac{\delta}{4n} \right)^{2n} \quad (\text{since } M \geq 4n^2\delta^{-1}) \\ &\geq 1 - 2n \frac{\delta}{4n} = 1 - \frac{\delta}{2} \end{aligned}$$

holds. Thus,

$$\frac{\Psi_n(1 + \frac{n}{M}, r(1 + \frac{n}{M}))}{\Psi_n(1, r)} \leq \frac{1}{1 - \frac{\delta}{2}} \leq 1 + \delta$$

for any $\delta \leq 1$, and we obtain the claim. \square

5 Hardness of the Volume of the Intersection of Two Cross-polytopes

This section establishes the following.

Theorem 5.1. *Given a vector $\mathbf{c} \in \mathbb{Z}_{>0}^n$ and integers $r_1, r_2 \in \mathbb{Z}_{>0}$, computing the volume of $C(\mathbf{0}, r_1) \cap C(\mathbf{c}, r_2)$ is #P-hard, even when each cross-polytopes contains the center of the other one, i.e., $\mathbf{0} \in C(\mathbf{c}, r_2)$ and $\mathbf{c} \in C(\mathbf{0}, r_1)$.*

The proof of Theorem 5.1 is a reduction of counting set partitions, which is a well-known #P-hard problem.

5.1 Idea for the reduction

To be precise, we reduce the following problem, which is a version of counting set partition.

Problem 1 (#LARGE SET). Given an integer vector $\mathbf{a} \in \mathbb{Z}_{>0}^n$ such that $\|\mathbf{a}\|_1$ is even, meaning that $\|\mathbf{a}\|_1/2$ is an integer, the problem is to compute

$$|\{\boldsymbol{\sigma} \in \{-1, 1\}^n \mid \langle \boldsymbol{\sigma}, \mathbf{a} \rangle > 0\}|. \quad (35)$$

Note that

$$|\{\boldsymbol{\sigma} \in \{-1, 1\}^n \mid \langle \boldsymbol{\sigma}, \mathbf{a} \rangle = 0\}| = \left| \left\{ S \subseteq \{1, \dots, n\} \mid \sum_{i \in S} a_i = \frac{\|\mathbf{a}\|_1}{2} \right\} \right|$$

holds: if $\boldsymbol{\sigma} \in \{-1, 1\}^n$ satisfies $\langle \boldsymbol{\sigma}, \mathbf{a} \rangle = 0$, then let $S \subseteq \{1, \dots, n\}$ be the set of indices of $\sigma_i = 1$ then $\sum_{i \in S} a_i = \|\mathbf{a}\|_1/2$ holds. Using the following simple observation, we see that Problem 1 is equivalent to counting set partitions.

Observation 5.2. *For any $\boldsymbol{\sigma} \in \{-1, 1\}^n$, $\langle \boldsymbol{\sigma}, \mathbf{a} \rangle > 0$ if and only if $\langle -\boldsymbol{\sigma}, \mathbf{a} \rangle < 0$.*

By Observation 5.2, we see that

$$|\{\boldsymbol{\sigma} \in \{-1, 1\}^n \mid \langle \boldsymbol{\sigma}, \mathbf{a} \rangle = 0\}| = 2^n - 2|\{\boldsymbol{\sigma} \in \{-1, 1\}^n \mid \langle \boldsymbol{\sigma}, \mathbf{a} \rangle > 0\}|. \quad (36)$$

In the following, let $\mathbf{a} \in \mathbb{Z}_{>0}^n$ be an instance of Problem 1. Roughly speaking, our proof of Theorem 5.1 claims that

$$\text{Vol}(C(\delta\mathbf{a}, 1) \cap C(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) \sim |\{\boldsymbol{\sigma} \in \{-1, 1\}^n \mid \langle \boldsymbol{\sigma}, \mathbf{a} \rangle > 0\}| \quad (37)$$

holds (see Figure 3), when $0 < \epsilon < \delta \ll 1/\|\mathbf{a}\|_1$. For convenience, we define

$$C_{\boldsymbol{\sigma}}(\mathbf{c}, r) = \{\mathbf{x} \in C(\mathbf{c}, r) \mid \sigma_i(x_i - c_i) \geq 0 \ (i = 1, \dots, n)\} \quad (38)$$

for any $\boldsymbol{\sigma} \in \{-1, 1\}^n$. Note that

$$C(\delta\mathbf{a}, 1) \cap C(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1) = \bigcup_{\boldsymbol{\sigma} \in \{-1, 1\}^n} C(\delta\mathbf{a}, 1) \cap C_{\boldsymbol{\sigma}}(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1) \quad (39)$$

holds. In the following, we claim for each $\boldsymbol{\sigma} \in \{-1, 1\}^n$ that

$$\text{Vol}(C(\delta\mathbf{a}, 1) \cap C_{\boldsymbol{\sigma}}(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) \simeq \begin{cases} 0 & (\text{if } \langle \boldsymbol{\sigma}, \mathbf{a} \rangle \leq 0) \\ \frac{\epsilon}{(n-1)!} & (\text{otherwise}) \end{cases}$$

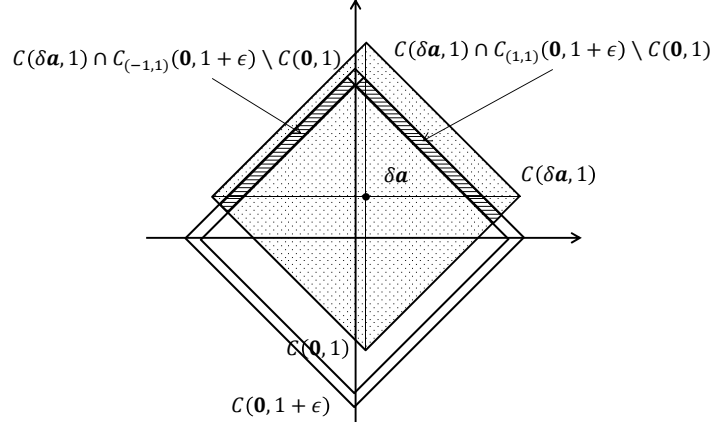


Figure 3: $C(0, 1 + \epsilon) \cap C(\delta \mathbf{a}, 1) \setminus C(0, 1)$.

with appropriate ϵ and δ .

First, we consider the case that $\boldsymbol{\sigma} \in \{-1, 1\}$ satisfies $\langle \boldsymbol{\sigma}, \mathbf{a} \rangle \leq 0$. We define

$$H_{\boldsymbol{\sigma}}^-(\mathbf{c}, r) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n \mid \langle \boldsymbol{\sigma}, \mathbf{x} - \mathbf{c} \rangle \leq r\} \quad (40)$$

$$H_{\boldsymbol{\sigma}}^+(\mathbf{c}, r) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n \mid \langle \boldsymbol{\sigma}, \mathbf{x} - \mathbf{c} \rangle > r\} \quad (41)$$

$$H_{\boldsymbol{\sigma}}(\mathbf{c}, r) \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n \mid \langle \boldsymbol{\sigma}, \mathbf{x} - \mathbf{c} \rangle = r\} \quad (42)$$

for convenience (see Figure 4).

5.2 Facet for $\langle \boldsymbol{\sigma}, \mathbf{a} \rangle \leq 0$

Proposition 5.3. *If $\langle \boldsymbol{\sigma}, \mathbf{a} \rangle \leq 0$, then $C(\delta \mathbf{a}, 1)$ is in the half-space $H_{\boldsymbol{\sigma}}^-(0, 1)$.*

Proof. Notice that $\mathbf{x} \in C(\delta \mathbf{a}, 1)$ implies

$$\langle \mathbf{x} - \delta \mathbf{a}, \boldsymbol{\sigma} \rangle \leq 1 \quad (43)$$

holds. Since the hypothesis that $\langle \mathbf{a}, \boldsymbol{\sigma} \rangle = 0$,

$$\langle \boldsymbol{\sigma}, \mathbf{x} - \delta \mathbf{a} \rangle = \langle \boldsymbol{\sigma}, \mathbf{x} \rangle - \delta \langle \boldsymbol{\sigma}, \mathbf{a} \rangle = \langle \boldsymbol{\sigma}, \mathbf{x} \rangle$$

holds, which implies with (43) that

$$\langle \boldsymbol{\sigma}, \mathbf{x} \rangle \leq 1. \quad (44)$$

We obtain the claim. \square

Proposition 5.3 implies $C_{\boldsymbol{\sigma}}(0, 1 + \epsilon) \setminus C(0, 1) \subset H_{\boldsymbol{\sigma}}^+(0, 1)$, and we see the following (see also Figure 5).

Corollary 5.4. *If $\langle \boldsymbol{\sigma}, \mathbf{a} \rangle \leq 0$, then $\text{Vol}(C(\delta \mathbf{a}, 1) \cap C_{\boldsymbol{\sigma}}(0, 1 + \epsilon) \setminus C(0, 1)) = 0$.*

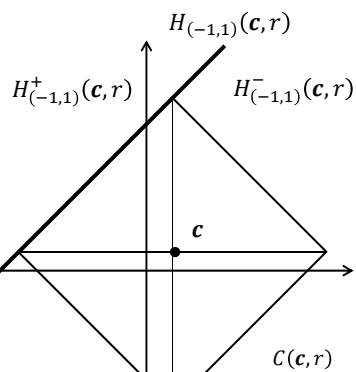


Figure 4: $H_{\sigma}^{-}(\mathbf{c}, r)$, $H_{\sigma}^{+}(\mathbf{c}, r)$, $H_{\sigma}(\mathbf{c}, r)$.

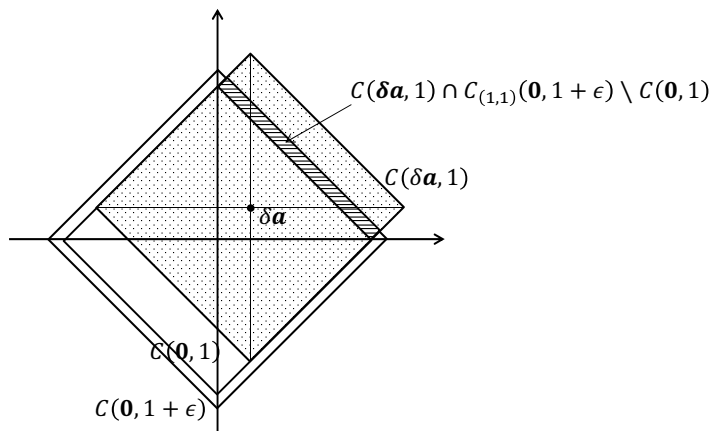


Figure 5: $C(\mathbf{0}, 1 + \epsilon) \cap C(\delta \mathbf{a}, 1) \setminus C(\mathbf{0}, 1)$ where $\langle \boldsymbol{\sigma}, \mathbf{a} \rangle = 0$ holds for $\boldsymbol{\sigma} = (1, -1)$ ($(-1, 1)$ as well).

5.3 Facet for $\langle \sigma, a \rangle > 0$

Next, we are concerned with the case that $\sigma \in \{-1, 1\}$ satisfies $\langle \sigma, a \rangle > 0$. Notice that $H_\sigma(\mathbf{0}, 1 + \epsilon)$ and $H_\sigma(\delta a, 1)$ are in parallel since they have a common normal vector σ .

Proposition 5.5. *Suppose that $\langle \sigma, a \rangle > 0$ holds. If $\epsilon < \delta$ then $H_\sigma^-(\mathbf{0}, 1 + \epsilon) \subseteq H_\sigma^-(\delta a, 1)$.*

Proof. Let $x \in H_\sigma^-(\mathbf{0}, 1 + \epsilon)$, then

$$\langle \sigma, x \rangle \leq 1 + \epsilon. \quad (45)$$

holds. Suppose for a contradiction that $x \notin H_\sigma^-(\delta a, 1)$, then

$$\langle \sigma, x - \delta a \rangle = \langle \sigma, x \rangle - \delta \langle \sigma, a \rangle > 1$$

holds. It implies

$$\langle \sigma, x \rangle > 1 + \delta \langle \sigma, a \rangle \geq 1 + \delta \quad (46)$$

holds since $\langle \sigma, a \rangle > 0$ means $\langle \sigma, a \rangle \geq 1$. Clearly, (46) and (45) contradict to the hypothesis that $\epsilon < \delta$. \square

Proposition 5.5 implies that $C(\delta a, 1) \cap C_\sigma(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1) \neq \emptyset$. More precisely, we observe the following, which we will use later.

Observation 5.6. *The L_2 distance between $H_\sigma(\mathbf{0}, 1 + \epsilon)$ and $H_\sigma(\mathbf{0}, 1)$ is $\frac{\epsilon}{\sqrt{n}}$.*

The volume of $C(\delta a, 1) \cap C_\sigma(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)$ is evaluated as follows, where we assume that ϵ is sufficiently small.

Proposition 5.7. *Suppose that $\langle \sigma, a \rangle > 0$ holds. If $\epsilon < \delta$ then*

$$\text{Vol}(C(\delta a, 1) \cap C_\sigma(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) \leq \frac{\epsilon}{(n-1)!} (1 + \epsilon)^{n-1}.$$

Proof.

$$\begin{aligned} \text{Vol}(C(\delta a, 1) \cap C_\sigma(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) &\leq \frac{1}{n!} ((1 + \epsilon)^n - 1) \\ &= \frac{\epsilon}{n!} \sum_{i=0}^{n-1} (1 + \epsilon)^i \\ &\leq \frac{\epsilon}{n!} n (1 + \epsilon)^{n-1} \\ &= \frac{\epsilon}{(n-1)!} (1 + \epsilon)^{n-1}. \end{aligned}$$

\square

Next, we give a lower bound of $\text{Vol}(C(\delta a, 1) \cap C_\sigma(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1))$. To begin with, we observe the following.

Observation 5.8. *For any vertex $v \in \{\pm e_1, \dots, \pm e_n\}$ of $C(\mathbf{0}, 1)$, the nearest vertex of $C(\delta a, 1)$ is in the L_2 distance $\delta \|a\|_2$.*

Observation 5.8 implies the following.

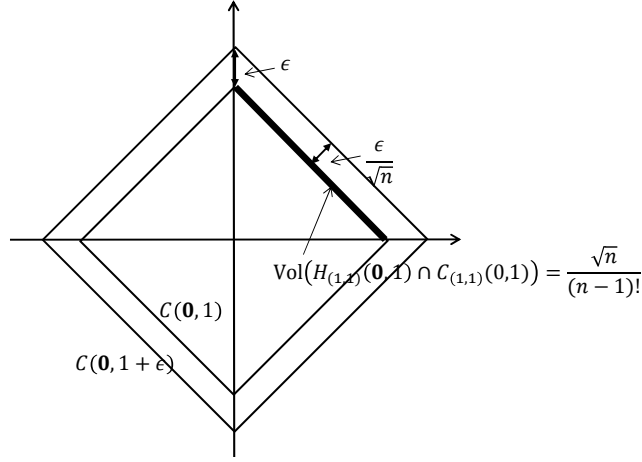


Figure 6: For Observation 5.6 and Proposition 5.10.

Proposition 5.9. For any vertex $v \in \{\pm e_1, \dots, \pm e_n\}$ of $C(\mathbf{0}, 1)$, one of hyperplanes $H_{\sigma}(\delta \mathbf{a}, 1)$ and $H_{-\sigma}(\delta \mathbf{a}, 1)$ is in the L_2 distance $\delta \|\mathbf{a}\|_2$ for any $\sigma \in \{-1, 1\}^n$. \square

Proposition 5.9 implies that when $\langle \sigma, \mathbf{a} \rangle > 0$, i.e., $C(\delta \mathbf{a}, 1) \cap C_{\sigma}(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1) \neq \emptyset$ holds, $H_{\sigma'}(\delta \mathbf{a}, 1)$ shaves off only a few area of $H_{\sigma}(\mathbf{0}, 1) \cap C(\delta \mathbf{a}, 1) \cap C_{\sigma}(\mathbf{0}, 1 + \epsilon)$. It is formally described as follows.

Proposition 5.10. When $\langle \sigma, \mathbf{a} \rangle > 0$,

$$\text{Vol}'(H_{\sigma}(\mathbf{0}, 1) \cap C_{\sigma}(\mathbf{0}, 1) \cap C(\delta \mathbf{a}, 1)) \geq \frac{\sqrt{n}}{(n-1)!} (1 - \delta \|\mathbf{a}\|_1)^{n-1}$$

holds, where $\text{Vol}'(S)$ denotes the $n-1$ dimensional volume of $S \subseteq \mathbb{R}^{n-1}$.

Proof. Remark that

$$\text{Vol}'(H_{\sigma}(\mathbf{0}, 1) \cap C_{\sigma}(\mathbf{0}, 1)) = \frac{\sqrt{n}}{(n-1)!}$$

since $\frac{1}{n} \frac{1}{\sqrt{n}} \text{Vol}'(H_{\sigma}(\mathbf{0}, 1) \cap C_{\sigma}(\mathbf{0}, 1)) = \text{Vol}(C_{\sigma}(\mathbf{0}, 1)) = \frac{1}{n!}$. Thus,

$$\begin{aligned} \text{Vol}'(H_{\sigma}(\mathbf{0}, 1) \cap C_{\sigma}(\mathbf{0}, 1) \cap C(\delta \mathbf{a}, 1)) &\geq \frac{\sqrt{n}}{(n-1)!} (1 - \delta \|\mathbf{a}\|_2)^{n-1} \\ &\geq \frac{\sqrt{n}}{(n-1)!} (1 - \delta \|\mathbf{a}\|_1)^{n-1} \end{aligned}$$

where the last inequality follows the fact $\|\mathbf{a}\|_2 \leq \|\mathbf{a}\|_1$. \square

Observation 5.6 and Proposition 5.10 implies the following.

Corollary 5.11. Suppose that $\langle \sigma, \mathbf{a} \rangle > 0$ holds. If $\epsilon < \delta$ then

$$\text{Vol}(C(\delta \mathbf{a}, 1) \cap C_{\sigma}(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) \geq \frac{\epsilon}{(n-1)!} (1 - \delta \|\mathbf{a}\|_1)^{n-1}.$$

5.4 Proof of Theorem 5.1

Proposition 5.12. *Suppose that $\langle \sigma, \mathbf{a} \rangle > 0$ holds. If $\epsilon < \delta$ and $\delta < \frac{0.1}{n2^n \|\mathbf{a}\|_1}$, then*

$$\text{Vol}(C(\delta \mathbf{a}, 1) \cap C_{\sigma}(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) \geq \frac{\epsilon}{(n-1)!} \left(1 - \frac{0.1}{2^n}\right).$$

Proof. By Corollary 5.11,

$$\begin{aligned} \text{Vol}(C(\delta \mathbf{a}, 1) \cap C_{\sigma}(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) &\geq \frac{\epsilon}{(n-1)!} (1 - \delta \|\mathbf{a}\|_1)^{n-1} \\ &\geq \frac{\epsilon}{(n-1)!} (1 - (n-1)\delta \|\mathbf{a}\|_1) \\ &\geq \frac{\epsilon}{(n-1)!} \left(1 - \frac{0.1}{2^n}\right). \end{aligned}$$

□

Now, we revisit the upper bound. When ϵ is small enough, Proposition 5.7 implies the following.

Proposition 5.13. *Suppose that $\langle \sigma, \mathbf{a} \rangle > 0$ holds. If $\epsilon < \delta$ and $\delta < \frac{0.1}{n2^n \|\mathbf{a}\|_1}$, then*

$$\text{Vol}(C(\delta \mathbf{a}, 1) \cap C_{\sigma}(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) \leq \frac{\epsilon}{(n-1)!} \left(1 + \frac{0.1}{2^n}\right).$$

Proof. Recall Proposition 5.7, which implies

$$\text{Vol}(C(\delta \mathbf{a}, 1) \cap C_{\sigma}(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) \leq \frac{\epsilon}{(n-1)!} (1 + \epsilon)^{n-1}$$

under the hypothesis. Remark that $\|\mathbf{a}\|_1 \geq n$ since $\mathbf{a} \in \mathbb{Z}_{>0}^n$. Thus $\epsilon < \frac{0.1}{n^2 2^n}$ is assumed by the hypothesis, and hence

$$\begin{aligned} \frac{\epsilon(1 + \epsilon)^{n-1}}{(n-1)!} &\leq \frac{\epsilon}{(n-1)!} \left(1 + \frac{0.1}{n^2 2^n}\right)^n \\ &\leq \frac{\epsilon}{(n-1)!} \left(1 + \frac{0.1}{2^n}\right). \end{aligned}$$

□

Corollary 5.4, Propositions 5.12 and 5.13 imply the following.

Lemma 5.14. *Suppose that $\epsilon < \delta$ and $\delta < \frac{0.1}{n2^n \|\mathbf{a}\|_1}$ hold. Let*

$$Z := \frac{\text{Vol}(C(\delta \mathbf{a}, 1) \cap C(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1))}{\frac{\epsilon}{(n-1)!}}$$

then

$$Z - 0.1 \leq |\{\sigma \in \{-1, 1\} \mid \langle \sigma, \mathbf{a} \rangle > 0\}| \leq Z + 0.1 \quad (47)$$

Proof. Proposition 5.12 implies that

$$\frac{\text{Vol}(C(\delta \mathbf{a}, 1) \cap C(\boldsymbol{\sigma}, 1 + \epsilon) \setminus C(\mathbf{0}, 1))}{\frac{\epsilon}{(n-1)!}} \geq 1 - \frac{0.1}{2^n}$$

holds. Clearly, $|\{\boldsymbol{\sigma} \in \{-1, 1\} \mid \langle \boldsymbol{\sigma}, \mathbf{a} \rangle > 0\}| \leq 2^n$, we obtain the lower bound of (47). The upper bound is similar. \square

Corollary 5.15. *Suppose that $\epsilon < \delta$ and $\delta < \frac{0.1}{n2^n \|\mathbf{a}\|_1}$ hold. Then,*

$$\left\lceil \frac{(n-1)!}{\epsilon} \text{Vol}(C(\delta \mathbf{a}, 1) \cap C(\mathbf{0}, 1 + \epsilon) \setminus C(\mathbf{0}, 1)) \right\rceil = |\{\boldsymbol{\sigma} \in \{-1, 1\} \mid \langle \boldsymbol{\sigma}, \mathbf{a} \rangle > 0\}|$$

where $[x]$ for $x \in \mathbb{R}$ denotes the integer z minimizing $|z - x|$.

To make values integer, set $\delta = 1/r$ and $\epsilon = \delta/2$, then we obtain the following.

Lemma 5.16. *Let $r = 10n2^n \|\mathbf{a}\|_1$, then*

$$[(n-1)! \text{Vol}(C(2\mathbf{a}, 2r) \cap C(\mathbf{0}, 2r+1) \setminus C(\mathbf{0}, 2r))] = |\{\boldsymbol{\sigma} \in \{-1, 1\} \mid \langle \boldsymbol{\sigma}, \mathbf{a} \rangle > 0\}|.$$

Finally, we remark that $\lg(r) = O(n \log n + \log \|\mathbf{a}\|_1)$, meaning that the reduction is in time polynomial in n and $\log \|\mathbf{a}\|_1$, which is the input size of Problem 1. Now we obtain Theorem 5.1.

6 Intersection of a Constant Number of Cross-polytopes

This section extends the algorithm in Section 4 to the intersection of k cross-polytopes for any *constant* $k \in \mathbb{Z}_+$. Let $\mathbf{p}_i \in \mathbb{R}^n$, $r_i \in \mathbb{R}_{\geq 0}$ and $C(\mathbf{p}_i, r_i)$ for $i = 1, \dots, k$, where $C(\mathbf{p}, r)$ is a cross-polytope (L_1 -ball) with center $\mathbf{p} \in \mathbb{R}^n$ and radius $r \in \mathbb{R}_{\geq 0}$. Then, we are to compute the following polytope given by

$$S(\Pi, \mathbf{r}) = \bigcap_{i=1}^k C(\mathbf{p}_i, r_i), \quad (48)$$

where Π is an $n \times k$ matrix $\Pi = (\mathbf{p}_1, \dots, \mathbf{p}_k)$ and $\mathbf{r} = (r_1, \dots, r_k)$. For the analysis, we assume that $\mathbf{p}_1, \dots, \mathbf{p}_k$ are internal points of $S(\Pi, \mathbf{r})$. Without loss of generality, we assume that $\mathbf{p}_1 = \mathbf{0}$ and $\mathbf{0} \leq \mathbf{r} \leq \mathbf{1}$.

We prove the following theorem.

Theorem 6.1. *There is an algorithm that outputs an approximation Z of $\text{Vol}(S(\Pi, \mathbf{r}))$ in $O(k^{k+2}n^{2k+3}/\delta^{k+1})$ time satisfying $\text{Vol}(S(\Pi, \mathbf{r})) \leq Z \leq (1 + \delta)\text{Vol}(S(\Pi, \mathbf{r}))$.*

6.1 Algorithm description

We explain the idea of our algorithm for approximating $\text{Vol}(S(\Pi, \mathbf{r}))$ as follows. First, $\text{Vol}(S(\Pi, \mathbf{r}))$ is given by the following probability

$$\text{Vol}(S(\Pi, \mathbf{r})) = 2^n \Pr \left[\bigwedge_{i=1}^k \|\mathbf{X} - \mathbf{p}_i\|_1 \leq r_i \right], \quad (49)$$

where $\mathbf{X} = (X_1, \dots, X_n)$ is a uniform random vector over $[-1, 1]^n$. We rewrite the probability as the repetition of an integral formula. Then, we staircase approximate the integral.

To transform (49) into the repetition of an integral formula, for Π and $\mathbf{u} \in \mathbb{R}^k$, we define

$$\Psi_j(\Pi, \mathbf{u}) = 2^j \Pr \left[\bigwedge_{i=1}^k \|\mathbf{X} - \mathbf{p}_i\|_1 \leq u_i \right], \quad (50)$$

so that we have $\text{Vol}(S(\Pi, \mathbf{r})) = \Psi_n(\Pi, \mathbf{r})$. We have $\Psi_0(\Pi, \mathbf{u}) = 1$ if $\mathbf{u} \geq \mathbf{0}$ and $\Psi_0(\Pi, \mathbf{u}) = 0$ otherwise. We can obtain $\Psi_j(\Pi, \mathbf{u})$ from $\Psi_{j-1}(\Pi, \mathbf{u})$ by

$$\Psi_j(\Pi, \mathbf{u}) = \int_{x_i \in [-1, 1]} \Psi_{j-1}(\Pi, \mathbf{u} - \mathbf{q}_j(x_j)) dx_j,$$

where $\mathbf{q}_j(x_j) = (|x_j - p_{1,j}|, \dots, |x_j - p_{k,j}|)$. Although this gives a simple expression for $S(\Pi, \mathbf{r})$, it is hard to compute the repetition of the integral because there are exponentially many breakpoints of the derivative of $\Psi_n(\Pi, \mathbf{u})$ of some order.

We compute the staircase approximation $G_j(\Pi, \mathbf{u})$ of $\Psi_j(\Pi, \mathbf{u})$ as follows. For convenience, we consider an intermediate $\overline{G}_j(\Pi, \mathbf{u})$ given by

$$\overline{G}_j(\Pi, \mathbf{u}) = \int_{s \in [-1, 1]} G_{j-1}(\Pi, \mathbf{u} - \mathbf{q}_j(s)) ds. \quad (51)$$

This integral can be reduced to a sum, which we will explain after we define $G_j(\Pi, \mathbf{u})$ for $j = 1, \dots, n$. After that, $G_j(\Pi, \mathbf{u})$ is a staircase approximation of $\overline{G}_j(\Pi, \mathbf{u})$ given by

$$G_j(\Pi, \mathbf{u}) = \overline{G}_j(\Pi, \lceil M\mathbf{u}/\mathbf{r} \rceil / M) \quad (52)$$

where $\lceil M\mathbf{u}/\mathbf{r} \rceil$ means a vector $(\lceil Mu_1/r_1 \rceil, \dots, \lceil Mu_k/r_k \rceil)$, and $M = 2kn^2/\delta$ is a parameter of our Algorithm 3 that is shown later. Note that the computation of $G_j(\Pi, \mathbf{u})$ is actually the computation of $(M+1)^k$ values. Since $\mathbf{u} - \mathbf{q}_j(s) \leq \mathbf{r}$ holds in the computation of (53) as long as $\mathbf{u} \leq \mathbf{r}$, we need not to have the value for the cases where $\mathbf{u} \leq \mathbf{r}$ does not hold.

Let us see that the integral for computing $\overline{G}_j(\Pi, \mathbf{u})$ can be transformed into a sum. We consider grid points Γ given by

$$\Gamma \stackrel{\text{def}}{=} \left\{ \frac{1}{M}(\ell_1 r_1, \ell_2 r_2, \dots, \ell_k r_k) \mid \ell_1, \dots, \ell_k \in \{0, 1, \dots, M\} \right\}.$$

For an arbitrary $\mathbf{u} \in \Gamma$, let

$$S_j(\mathbf{u}) \stackrel{\text{def}}{=} \{s \in [-1, 1] \mid \exists i \in \{1, \dots, k\}, \exists \ell \in \mathbb{Z} \text{ s.t. } u_j - |s - p_{i,j}| = \ell r_i / M\},$$

for $j = 1, \dots, k$. Then let

$$T_j(\mathbf{u}) \stackrel{\text{def}}{=} \bigcup_{j=1}^k S_j(\mathbf{u}) \cup \{-1, 1\}.$$

Suppose t_0, t_1, \dots, t_m be an ordering of all elements of $T_i(\mathbf{u}, v)$ such that $t_i \leq t_{i+1}$ for any $i = 0, 1, \dots, m$. Then we can compute $G_i(\Pi, \mathbf{u})$ for any $\mathbf{u} \in \Gamma$ by

$$\begin{aligned} G_j(\Pi, \mathbf{u}) &= \overline{G}_j(\Pi, \mathbf{u}) \\ &= \int_{-1}^1 G_{j-1}(\Pi, \mathbf{u} - \mathbf{q}_j(s)) ds \\ &= \sum_{i=0, \dots, m} \int_{t_i}^{t_{i+1}} G_{j-1} \left(\Pi, \frac{1}{M} \mathbf{w}(\mathbf{u}, t_{i+1}) \right) ds && \text{(by (52))} \\ &= \sum_{i=0, \dots, m} (t_{i+1} - t_i) G_{j-1} \left(\Pi, \frac{1}{M} \mathbf{w}(\mathbf{u}, t_{i+1}) \right), && (53) \end{aligned}$$

where $w(\mathbf{u}, t_{i+1}) = (\lceil M(u_1 - |t_{i+1} - p_{1,j}|) \rceil, \dots, \lceil M(u_k - |t_{i+1} - p_{k,j}|) \rceil)$.

Our algorithm outputs the value of $G_n(\Pi, \mathbf{r})$. By taking the parameter M larger, we get closer approximation of $\text{Vol}(S(\Pi, \mathbf{r}))$. Here we assume that $0 < \delta \leq 1/2$. The following is our algorithm 3.

Algorithm 3. Input: $\Pi \in \mathbb{R}^{kn}$, $\mathbf{0} \leq \mathbf{r} \leq \mathbf{1}$;

1. Let $M := 2kn^2/\delta$;
2. Let $G_0(\Pi, \mathbf{u}) := 1$ for $\mathbf{u} \geq \mathbf{0}$, otherwise $G_0(\Pi, \mathbf{u}) := 0$;
3. For $j := 1, \dots, n$,
4. Compute $\overline{G}_j(\Pi, \mathbf{u})$ from $G_{j-1}(\Pi, \mathbf{u})$ by (53);
5. Compute staircase approximation $G_j(\Pi, \mathbf{u})$ of $\overline{G}_j(\Pi, \mathbf{u})$ by (52);
6. Output $G_n(\Pi, \mathbf{r})$.

Let us consider the running time of our algorithm 3. In Step 4-5, computing $\overline{G}_j(\Pi, \mathbf{u})$ for a fixed \mathbf{u} takes $O(kM)$ time because $\overline{G}_j(\Pi, \mathbf{u})$ is the sum of $m \leq 2kM$ values. We compute $\overline{G}_j(\Pi, \mathbf{u})$ for $(M+1)^k$ different \mathbf{u} 's. Then Step 4-5 is repeated n times. We have the following observation.

Observation 6.2. The running time of Algorithm 3 is $O(knM^{k+1})$.

6.2 Proof of Theorem 6.1

Here, we prove that $M = 2kn^2/\delta$ is sufficient to have $1 + \delta$ approximation of $\text{Vol}(S(\Pi, \mathbf{r}))$. We show the following lemma.

Lemma 6.3. $\Psi_j(\Pi, \mathbf{u})$ is non-decreasing with respect to each component of \mathbf{u} .

Proof. Let $\mathbf{u} = (u_1, \dots, u_k) \leq \mathbf{u}' = (u'_1, \dots, u'_k)$. By definition, we have that

$$\begin{aligned} \Psi_j(\Pi, \mathbf{u}) &= \text{Vol} \left(\left\{ \mathbf{x} \in \mathbb{R}^j \mid \bigwedge_{i=1}^k \sum_{\ell=1}^j |x_\ell - p_{i,\ell}| \leq u_i \right\} \right) \\ &\leq \text{Vol} \left(\left\{ \mathbf{x} \in \mathbb{R}^j \mid \bigwedge_{i=1}^k \sum_{\ell=1}^j |x_\ell - p_{i,\ell}| \leq u'_i \right\} \right) = \Psi_j(\Pi, \mathbf{u}'). \end{aligned}$$

□

Then, we can prove the following lemma, which gives upper and lower bounds on the approximation $G_n(\Pi, \mathbf{u})$.

Lemma 6.4. $\Psi_n(\Pi, \mathbf{u}) \leq G_n(\Pi, \mathbf{u}) \leq \Psi_n(\Pi, \mathbf{u} + n\mathbf{r}/M)$.

Proof. Since $\Psi_n(\Pi, \mathbf{u}) \leq G_n(\Pi, \mathbf{u})$ is clear from the algorithm, we prove $G_n(\Pi, \mathbf{u}) \leq \Psi_n(\Pi, \mathbf{u} + n\mathbf{r}/M)$ in the following. This is proved by induction on n . Since $G_0(\Pi, \mathbf{u}) = \Psi_0(\Pi, \mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}_{\geq 0}^k$, the base case holds. Then, as for the induction step, we assume $G_j(\Pi, \mathbf{u}) \leq \Psi_j(\Pi, \mathbf{u} + j\mathbf{r}/M)$. By the definition of $G_j(\Pi, \mathbf{u})$ and $\overline{G}_{j+1}(\Pi, \mathbf{u})$, we have

$$\begin{aligned} G_{j+1}(\Pi, \mathbf{u}) &= \overline{G}_{j+1}(\Pi, \lceil M\mathbf{u}/\mathbf{r} \rceil / M) \\ &= \int_{-1}^1 G_j(\Pi, \lceil M\mathbf{u}/\mathbf{r} \rceil / M - \mathbf{q}_{j+1}(s)) ds \\ &\leq \int_{-1}^1 \Psi_j(\Pi, \lceil M\mathbf{u}/\mathbf{r} \rceil / M - \mathbf{q}_{j+1}(s) + j\mathbf{r}/M) ds && \text{(Induction hypo.)} \\ &\leq \int_{-1}^1 \Psi_j(\Pi, \mathbf{u}/\mathbf{r} - \mathbf{q}_{j+1}(s) + (j+1)\mathbf{r}/M) ds \\ &= \Psi_{j+1}(\Pi, \mathbf{u}/\mathbf{r} + (j+1)\mathbf{r}/M), \end{aligned}$$

where $\mathbf{u}/\mathbf{r} = (u_1/r_1, \dots, u_k/r_k)$ and $\lceil \mathbf{u}/\mathbf{r} \rceil = (\lceil u_1/r_1 \rceil, \dots, \lceil u_k/r_k \rceil)$. Then we have the lemma. \square

We prove Theorem 6.1 as follows.

Proof of Theorem 6.1: By Lemma 6.4, we have that the approximation ratio is bounded from above by $\Psi_n(\Pi, \mathbf{u} + \mathbf{h})/\Psi_n(\Pi, \mathbf{u})$, where $\mathbf{h} = (h_1, \dots, h_k) \leq n\mathbf{r}/M$. We bound the reciprocal of the approximation ratio from below.

For convenience, let

$$K_i(\Pi, \mathbf{u}, d) = \{ \mathbf{x} \in \mathbb{R}^n \mid \exists \mathbf{y} \in S(\Pi, \mathbf{u}), \exists b \in [0, d], \mathbf{x} - \mathbf{p}_i = b(\mathbf{y} - \mathbf{p}_i), \text{ s.t. } \|\mathbf{y} - \mathbf{p}_i\|_1 = u_i \}.$$

Here $K_i(\Pi, \mathbf{u}, 1)$ is given by considering the cones that are given by the center \mathbf{p}_i as the top vertex and the shared surface of $S(\Pi, \mathbf{u})$ and $C(\mathbf{p}_i, u_i)$ as the bottom. Then $K_i(\Pi, \mathbf{u}, d)$ is given by scaling $K_i(\Pi, \mathbf{u}, 1)$. Since we assume that $\mathbf{0} \in S(\Pi, \mathbf{u})$, we have $K_i(\Pi, \mathbf{u}, 1) \subseteq S(\Pi, \mathbf{u})$. Since $\text{Vol}(S(\Pi, \mathbf{u}) - K_i(\Pi, \mathbf{u}, 1))$ is equal to $\text{Vol}(S(\Pi, \mathbf{u} + h_i \mathbf{e}_i) - K_i(\Pi, \mathbf{u}, (u_i + h_i)/u_i))$, we have that

$$\begin{aligned} \frac{\Psi_n(\Pi, \mathbf{u})}{\Psi_n(\Pi, \mathbf{u} + h_i \mathbf{e}_i)} &= \frac{\text{Vol}(S(\Pi, \mathbf{u}))}{\text{Vol}(S(\Pi, \mathbf{u} + h_i \mathbf{e}_i))} \\ &= \frac{\text{Vol}(S(\Pi, \mathbf{u}) - K_i(\Pi, \mathbf{u}, 1)) + \text{Vol}(K_i(\Pi, \mathbf{u}, 1))}{\text{Vol}(S(\Pi, \mathbf{u} + h_i \mathbf{e}_i) - K_i(\Pi, \mathbf{u}, (u_i + h_i)/u_i)) + \text{Vol}(K_i(\Pi, \mathbf{u}, (u_i + h_i)/u_i))} \\ &\geq \frac{\text{Vol}(K_i(\Pi, \mathbf{u}, 1))}{\text{Vol}(K_i(\Pi, \mathbf{u}, (u_i + h_i)/u_i))} \geq \frac{1}{(1 + h_i/u_i)^n}. \end{aligned}$$

This leads to

$$\frac{\Psi_n(\Pi, \mathbf{r})}{\Psi_n(\Pi, \mathbf{r} + \mathbf{h})} \geq \prod_{i=1}^k \frac{1}{(1 + h_i/r_i)^n} \geq 1 - \sum_{i=1}^k nh_i/r_i.$$

Then, for $\delta \leq 1/2$, we have $\frac{\Psi_n(\Pi, \mathbf{r} + \mathbf{h})}{\Psi_n(\Pi, \mathbf{r})} \leq \frac{1}{1 - \sum_{i=1}^k nh_i/r_i} \leq \frac{1}{1 - kn^2/M} \leq 1 + 2kn^2/M = 1 + \delta$. \square

7 The Volume of \mathcal{V} -polytopes with $n + k$ Vertices

Given a vertex set $V = \{\mathbf{v}_1, \dots, \mathbf{v}_{n+k}\}$, where $k \geq 1$ is a constant. Here we consider the problem of computing the volume of $P = \text{conv}(V)$. Without loss of generality, we assume that P contains the origin $\mathbf{0}$ as its interior point. Also note that we assume that all the vectors are vertical vectors. Then we have the following Theorem.

Theorem 7.1. *By decomposing P into simplices, we can compute $\text{Vol}(P)$ in $O(n^{k+3})$ time.*

The following is the algorithm for computing $\text{Vol}(P)$. For all possible $U \subseteq V$, we check if the $n - 1$ dimensional polytope f_U given by U is the facet of P , and if so, we compute the volume $S_U := \det(M_U)/n!$, where $M_U = (\mathbf{u}_1, \dots, \mathbf{u}_n)$. Then $\text{Vol}(P) = \sum_{U \subseteq V} S_U$.

Algorithm 4. Input: $V = \{\mathbf{v}_1, \dots, \mathbf{v}_{n+k}\} \in \mathbb{R}^{n(n+k)}$

1. $S := 0$, $M_V := (\mathbf{v}_1, \dots, \mathbf{v}_{n+k})$;
2. For all possible $U = \{\mathbf{u}_1, \dots, \mathbf{u}_n\} \subseteq V$,
3. Compute $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ s.t. $\mathbf{a}^\top M_U = \mathbf{1}$, where $M_U = (\mathbf{u}_1, \dots, \mathbf{u}_n)$;
4. If $\mathbf{a}^\top M_V \leq \mathbf{1}$ or $\mathbf{a}^\top M_V \geq \mathbf{1}$, then
5. $S := S + \text{Vol}(S_U)$, where $\text{Vol}(S_U) = \det(M_U)/n!$;
6. Output S .

We consider the running time of the algorithm. The loop from Step 2 to Step 5 is repeated $\binom{n+k}{k}$ times. In Step 3, we compute \mathbf{a} by the Gaussian elimination, which takes $O(n^3)$ time. Step 4 checks if all vertices is contained in a half space given by f_U . This takes at most $n(n+k)$ additions and multiplications. In Step 5, computing $\text{Vol}(S_U)$ takes $O(n^3)$. The running time amounts to $O\left((n^3 + n(n+k) + n^3)\binom{n+k}{k}\right) = O(n^{k+3})$.

8 Conclusion

Motivated by a deterministic approximation of the volume of a \mathcal{V} -polytope, this paper gave an FPTAS for the volume of the knapsack dual polytope $\text{Vol}(P_{\mathbf{a}})$. In the process, we showed that the volume of the intersection of L_1 -balls is #P-hard, and gave an FPTAS. As we remarked, the volume of the intersection of two L_q -balls are easy for $q = 2, \infty$. The complexity of the volume of the intersection of two L_q -balls for other $q > 0$ is interesting. The problem seems difficult even for approximation in the case of $q \in (0, 1)$, since L_q -ball is no longer convex. Our FPTAS for the intersection of two cross-polytopes assumes that each cross-polytope contains the center of the other one. It is open if an FPATS exists without the assumption.

We have remarked that the volume of a \mathcal{V} -polytope with $n+k$ vertex is computed in $O(n^{k+3})$, while Khachiyan's result [14] implies that it is #P-hard when $k \geq n+1$. The complexity when $k = \omega(1)$ and $k = o(n)$ seems not known. It is an interesting question if an FPT algorithm regarding k exists.

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